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Online Publication Date: 01 January 1997
To cite this Article: Wang, Bo-Ying and Zhang, Fuzhen (1997) ‘Schur complements and matrix inequalities of Hadamard products’, Linear and Multilinear Algebra, 43:1, 315 - 326
To link to this article: DOI: 10.1080/03081089708818531
URL: http://dx.doi.org/10.1080/03081089708818531
Schur Complements and Matrix Inequalities of Hadamard Products*

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Communicated by R. Grone

In memory of Bob Thompson

(Received 7 October 1996; In final form 15 July 1997)

Let $A$ and $B$ be $n$-square positive definite matrices. Denote the Hadamard product of $A$ and $B$ by $A \circ B$. The main results of the paper are:

1. For any matrices $C$ and $D$ of size $m \times n$

$$(C \circ D)(A \circ B)^{-1}(C \circ D)^* \leq (CA^{-1}C^*) \circ (DB^{-1}D^*)$$

and

2. Let $A/\alpha$ be the Schur complement of $A(\alpha)$ in $A$. Then

$$(A \circ B)/\alpha \geq A/\alpha \circ B/\alpha.$$  Some other matrix inequalities of Schur complements and Hadamard products of positive definite matrices are also presented.

Keywords: Schur complement; Hadamard product; principal submatrix

*AMS Subject: 15A45

1The work was supported in part by an NSF grant of China.
*The work was supported in part by the Nova Faculty Development Funds.
1. INTRODUCTION

Let $\alpha$ and $\beta$ be proper index subsets of \{1, 2, \ldots, n\}. For any $n$-square complex matrix $A$, denote by $A(\alpha, \beta)$, or simply $A(\alpha)$ if $\alpha = \beta$, the submatrix of $A$ lying in rows $\alpha$ and columns $\beta$. Let $\alpha'$ be the complement of $\alpha$ in \{1, 2, \ldots, n\}.

In $A(\alpha)$ is nonsingular, the matrix

$$A(\alpha') = A(\alpha', \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha'),$$

is called the Schur complement of $A(\alpha)$ in $A$, and is denoted by $A/\alpha$.

It is well known [HJ, p. 221] that

$$\det A = \det A(\alpha) \det (A/\alpha).$$

In addition, there exists a permutation matrix $P$ such that

$$P^TAP = \begin{pmatrix} A(\alpha) & A(\alpha, \alpha') \\ A(\alpha', \alpha) & A(\alpha') \end{pmatrix}$$

In particular, if $\alpha = \{1, 2, \ldots, k\}$ and if $A$ is partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11}$ is the leading $k \times k$ principal submatrix of $A$, then

$$A/\alpha = A_{22} - A_{21}A_{11}^{-1}A_{12}.$$  

We will adopt the notation $\widetilde{A}_{11}$ in place of $A/\alpha$ when the Schur complement of the (1, 1)-block in $A$ is concerned. Similarly $\widetilde{A}_{22}$ means the Schur complement of the (2, 2)-block in $A$. Since $(P^TAP)_{11} = A(\alpha)$, we will state our theorems in terms of general form $A/\alpha$ and provide proofs for $A_{11}$. Permutation similarity implies the results for the general case of $A/\alpha$.

This paper deals with the inequalities of the Hadamard products of the Schur complements of positive definite matrices.
As usual, we write $A > 0 \ (\geq 0)$ if $A$ is positive (semi-)definite, and $A \leq B$ or $B \geq A$ if $B - A \geq 0$ for positive semidefinite matrices $A$ and $B$. A useful fact is that

$$0 < A \leq B \Rightarrow B^{-1} \leq A^{-1}.$$ 

Let $A \succeq 0$ be partitioned as, with $A_{21} = A_{12}^T,$

$$
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}.
$$

If $A_{11}$ is nonsingular, then $A_{11} \geq 0$ since

$$
\begin{pmatrix}
I & 0 \\
-A_{21}A_{11}^{-1} & I
\end{pmatrix}
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\begin{pmatrix}
I & -A_{11}^{-1}A_{12} \\
0 & I
\end{pmatrix}
= 
\begin{pmatrix}
A_{11} & 0 \\
0 & A_{11}
\end{pmatrix}
$$

It follows that if $A > 0$ then

$$A(\alpha') \geq A/\alpha \quad (1)$$

It is well known [HJ, p. 18, or Zh, p. 54] that $A^{-1}$ takes the form

$$A^{-1} = \begin{pmatrix}
\bar{A}_{22}^{-1} & * \\
* & \bar{A}_{11}^{-1}
\end{pmatrix}, \quad (2)$$

where $*$ denotes entries irrelevant to our discussions; that is,

$$(A^{-1})_{11} = \bar{A}_{22}^{-1}; \ (A^{-1})_{22} = \bar{A}_{11}^{-1} \quad (3)$$

or more generally

$$A^{-1}(\alpha) = (A/\alpha')^{-1} \quad (4)$$

Furthermore, by writing $A = (A^{-1})^{-1}$, we have from (2) that

$$\bar{A}_{11}^{-1}^{-1} = A_{11} \quad \text{or} \quad \bar{A}_{11}^{-1} = A_{11}^{-1}. \quad (5)$$

Note that the second Schur complements were taken in $A^{-1}$. We will not mention the underlying matrix if no confusion arises.
It is immediate from the Fischer's determinantal inequality and (2) that
\[ \det \overline{A_{11}} \det \overline{A_{22}} \leq \det A \leq \det A_{11} \det A_{22}, \text{ if } A > 0. \]

It has been shown [H, FM] that if \( A > 0 \) and \( B > 0 \) are of the same size and are partitioned conformally, then
\[ A_{11} \overline{+} B_{11} \geq A_{11} + B_{11}, \] (6)
or equivalently,
\[ (A + B)/\alpha \geq A/\alpha + B/\alpha. \]

It follows more generally that for any positive numbers \( s \) and \( t, \)
\[ (sA + tB)/\alpha \geq sA/\alpha + tB/\alpha. \]

Recall that the Hadamard product of \( A \) and \( B \) with the same size, denoted by \( A \circ B \), is the entrywise product \( A \circ B = (a_{ij}b_{ij}) \).
An interesting result due to I. Schur is
\[ A \geq 0, B \geq 0 \Rightarrow A \circ B \geq 0. \] (7)

The strict inequality on the right hand side of (7) holds when \( A \) and \( B \) are both positive definite matrices.
We end this section by noting that if \( A > 0 \), then, from (1) and (4),
\[ A^{-1}(\alpha) - [A(\alpha)]^{-1} \geq 0, \] (8)
which is shown by Chollet in [C]. Chollet's proof is inductive by first dealing with the case of principal submatrices of size \( n-1 \), using the Jacobi determinant theorem.

2. ON HADAMARD PRODUCT

Utilizing the Schur complement gives the following result.
**Theorem 1** Let $A$ and $B$ be $n$-square positive definite matrices, and let $C$ and $D$ be any matrices of size $m \times n$. Then

$$ (C \circ D)(A \circ B)^{-1}(C \circ D)^* \leq (CA^{-1}C^*) \circ (DB^{-1}D^*). $$

In particular

$$ (A \circ B)^{-1} \leq A^{-1} \circ B^{-1} $$

and

$$ (C \circ D)(C \circ D)^* \leq (CC^*) \circ (DD^*). $$

**Proof** Let

$$ \hat{A} = \begin{pmatrix} A & C^* \\ C & CA^{-1}C^* \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} B & D^* \\ D & DB^{-1}D^* \end{pmatrix}. $$

It is immediate that $\hat{A} \geq 0$ and $\hat{B} \geq 0$. By (7), we have $A \circ \hat{B} \geq 0$; that is,

$$ \begin{pmatrix} A \circ B & C^* \circ D^* \\ C \circ D & (CA^{-1}C^*) \circ (DB^{-1}D^*) \end{pmatrix} \geq 0. $$

Taking the Schur complement of the $(1,1)$-block, we have

$$ (CA^{-1}C^*) \circ (DB^{-1}D^*) - (C \circ D)(A \circ B)^{-1}(C \circ D)^* \geq 0. $$

Putting $C = D = I$ and $A = B = I$, respectively, gives (10) and (11).

Note that (10) is also immediate from (8), for

$$ (A \otimes B)^{-1} = A^{-1} \otimes B^{-1} $$

and $A \circ B$ is a principal submatrix of the tensor product $A \otimes B$.

**Theorem 2** Let $A > 0$ and $B > 0$ be of the same size. Then

$$ (A \circ B)/\alpha \geq A/\alpha \circ B/\alpha. $$

Equivalently

$$ A_{11} \circ B_{11} \geq \widetilde{A_{11}} \circ \widetilde{B_{11}}. $$
Proof Partition $A$ and $B$ conformally as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}. $$

Then

$$A \circ B = \begin{pmatrix} A_{11} \circ B_{11} & A_{12} \circ B_{12} \\ A_{21} \circ B_{21} & A_{22} \circ B_{22} \end{pmatrix}. $$

Taking the Schur complement of the $(1,1)$-block in $A \circ B$, one has

$$(A \circ B)_{11} = A_{22} \circ B_{22} - (A_{21} \circ B_{21})(A_{11} \circ B_{11})^{-1}(A_{12} \circ B_{12}).$$

By (9) and noticing that

$$A_{21} A_{11}^{-1} A_{12} = A_{22} - A_{11} \geq 0, \quad B_{21} B_{11}^{-1} B_{12} = B_{22} - B_{11} \geq 0,$$

one gets

$$(A \circ B)_{11} \geq A_{11} \circ B_{22} + (A_{22} - A_{11}) \circ B_{11}$$

$$\geq A_{11} \circ B_{22}$$

$$\geq A_{11} \circ B_{11}. \quad \square$$

It is seen from the proof that

$$(A \circ B)/\alpha \geq A/\alpha \circ B'/\alpha + [A(\alpha') - A/\alpha] \circ B/\alpha \geq A \circ \alpha \circ B(\alpha').$$

(13)

For more results, we borrow the following from [A] or [WZ]:

If $A$ is positive definite, then for any integer $m$

$$A(\alpha) \leq [A^m(\alpha)]^{1/2} \quad \text{if} \quad m > 0, \quad \text{and} \quad A(\alpha) \geq [A^m(\alpha)]^{1/2} \quad \text{if} \quad m < 0.$$  

(14)

Theorem 3 Let $A > 0$ and let $m$ be any integer. Then

$$(A^m/\alpha)^{1/2} \leq A/\alpha \quad \text{if} \quad m > 0, \quad \text{and} \quad (A^m/\alpha)^{1/2} \geq A/\alpha \quad \text{if} \quad m < 0.$$  

(15)
In particular

\[ A^{-1}/\alpha \leq (A/\alpha)^{-1}. \] \hfill (16)

**Proof**  To show (15), let

\[ A = \begin{pmatrix} A_{11} & \ast \\ \ast & A_{22} \end{pmatrix} \quad \text{and} \quad A_m = \begin{pmatrix} B_{11} & \ast \\ \ast & B_{22} \end{pmatrix}. \]

Then the identity \((A^m)^{-1} = (A^{-1})^m\) yields, by (2),

\[ \begin{pmatrix} B_{22}^{-1} & \ast \\ \ast & B_{11}^{-1} \end{pmatrix} = \begin{pmatrix} A_{22}^{-1} & \ast \\ \ast & A_{11}^{-1} \end{pmatrix}^m. \]

It follows from (14) that if \(m > 0\) then

\[ \widetilde{A}_{11}^{-1} \leq (\widetilde{B}_{11}^{-1})^{1/2} \quad \text{or} \quad \widetilde{B}_{11}^{-1} \leq \widetilde{A}_{11}, \]

and that if \(m < 0\) then the direction of the inequality is reversed. (16) is obtained by setting \(m = -1\). 

(16) is also an immediate consequence of (8) by using (4) or from the argument as follows: by (8),

\[ A_{22}^{-1} \leq (A^{-1})_{22} \quad \text{or} \quad A_{22}^{-1} \leq \widetilde{A}_{11}^{-1}. \]

On the other hand, writing \(A\) as the inverse of \(A^{-1}\), we have

\[ A_{22}^{-1} = (A^{-1})_{11}. \]

Therefore

\[ (\widetilde{A^{-1}})_{11} \leq \widetilde{A}_{22}^{-1}, \]

which is the desired inequality.

**Corollary 1**  Let \(A > 0\) and \(B > 0\) be of the same size. Then

\[ (A \circ B)^{-1}/\alpha \leq [(A \circ B)/\alpha]^{-1} \leq (A/\alpha \circ B/\alpha)^{-1} \leq (A/\alpha)^{-1} \circ (B/\alpha)^{-1} \] \hfill (17)
and

\[(A \circ B)^{-1} - \alpha \leq (A^{-1} - \alpha) \circ (B^{-1} - \alpha) \leq (A^{-1} \circ B^{-1}) - \alpha \leq (A^{-1} - \alpha) \circ (B^{-1})^{-1} \]

(18)

**Proof** The inequalities in (17) are consequences of inequalities (16), (12) and (10), respectively.

For the first inequality of (18), notice that the inequality

\[(A_{22} \circ B_{22})^{-1} \leq A_{22}^{-1} \circ B_{22}^{-1} \]

is equivalent to, by (5),

\[(\widetilde{A_{22}} \circ \widetilde{B_{22}})^{-1} \leq \widetilde{A_{22}}^{-1} \circ \widetilde{B_{22}}^{-1} \]

or, by (3),

\[((A \circ B)^{-1})_{11} \leq (A^{-1})_{11} \circ (B^{-1})_{11}, \]

which gives the first inequality of (18).

The second inequality of (18) follows from (12).

To show the last inequality of (18), we partition \(A\) and \(B\) as in the proof of Theorem 2. Then \(A^{-1} \circ B^{-1}\) takes the form

\[A^{-1} \circ B^{-1} = \begin{pmatrix} X & Y \\ Y^* & A_{11}^{-1} \circ B_{11}^{-1} \end{pmatrix}, \]

where \(X\) and \(Y\) are some matrices. It follows that

\[(A^{-1} \circ B^{-1})_{11} = A_{11}^{-1} \circ B_{11}^{-1} - Y^* X^{-1} Y \leq A_{11}^{-1} \circ B_{11}^{-1}, \]

which is the desired inequality.

**Corollary 2** Let \(A > 0\) and \(B > 0\) be of the same size. Then

\[((A \circ B)/\alpha)^{-1} \leq (A^{-1} \circ B^{-1})/\alpha. \]

(19)
Proof Compute

\[
\left[(A \odot B)/\alpha\right]^{-1} \leq \left[A/\alpha \circ B(\alpha')\right]^{-1} \quad \text{(by (13))}
\]
\[
= \left[A/\alpha \circ (B^{-1}/\alpha)^{-1}\right]^{-1} \quad \text{(by (4))}
\]
\[
\leq (A/\alpha)^{-1} \circ \left(B^{-1}/\alpha\right) \quad \text{(by (10))}
\]
\[
\leq (A/\alpha)^{-1} \circ \left(B^{-1}/\alpha\right) + A^{-1}/\alpha \circ \left[(B/\alpha)^{-1} - B^{-1}/\alpha\right] \quad \text{(by (16))}
\]
\[
\leq \left(A^{-1} \circ B^{-1}\right)/\alpha \quad \text{(by (13) and (4))}
\]

Remark 2.1 Inequalities (10) and (11) have appeared in the literatures; see, for instance, [A], [Ho], or [Z].

Remark 2.2 All other comparisons of the terms in (17), (18) and (19) are impossible. For instance, \((A^{-1}/\alpha) \circ \left(B^{-1}/\alpha\right)\) and \((A/\alpha \circ B/\alpha)^{-1}\) are not comparable. Take \(\alpha = \{1, 2\}\) and

\[
A = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
0 & 1 & 2 & 1 \\
0 & 0 & 1 & 2
\end{pmatrix}, \quad B = \begin{pmatrix}
4 & 1 & 0 & 1 \\
1 & 2 & 1 & 0 \\
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 5
\end{pmatrix}
\]

Then the eigenvalues of \((A^{-1}/\alpha) \circ \left(B^{-1}/\alpha\right) - (A/\alpha \circ B/\alpha)^{-1}\) are \(-2.7069\) and 0.1024.

3. ON PRINCIPAL SUBMATRICES

We compare in this section the principal submatrices of sum and Hadamard product to those of the inverses, and provide examples when comparisons are impossible.

Since for \(A > 0\) and \(B > 0\) of the same size,

\[
(A + B)^{-1} \leq A^{-1} \quad \text{and} \quad (A + B)^{-1} \leq B^{-1},
\]

we have (in fact \(\frac{1}{2}\) can be replaced by \(\frac{1}{2}\))

\[
(A + B)^{-1} \leq \frac{1}{2} (A^{-1} + B^{-1}) \leq A^{-1} + B^{-1}.
\]
It follows that
\[(A + B)^{-1}(\alpha) \leq A^{-1}(\alpha) + B^{-1}(\alpha)\]  \hspace{1cm} (20)
and
\[
[(A + B)(\alpha)]^{-1} \leq [A(\alpha)]^{-1} + [B(\alpha)]^{-1}.
\]  \hspace{1cm} (21)

For the left hand sides of (20) and (21), we have, by (8),
\[
[(A + B)(\alpha)]^{-1} \leq (A + B)^{-1}(\alpha).
\]

Similarly for the right hand sides,
\[
[A(\alpha)]^{-1} + [B(\alpha)]^{-1} \leq A^{-1}(\alpha) + B^{-1}(\alpha).
\]

But the left hand side of (20) and the right hand side of (21) not comparable.

To see this, let \(\alpha = \{1\}\) and take
\[
A = B = \begin{pmatrix} 1 & 1 \\ 1 & 1.1 \end{pmatrix}, \quad A = B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},
\]

respectively. One gets inequalities in both directions.

Using (5) we write
\[
A_{11}^{-1} + B_{11}^{-1} = A_{11}^{-1} + B_{11}^{-1}.
\]

By (6) we have
\[
A_{11}^{-1} + B_{11}^{-1} \leq A_{11}^{-1} + B_{11}^{-1},
\]
or
\[
(A_{11}^{-1} + B_{11}^{-1})^{-1} = [(A^{-1} + B^{-1})^{-1}]_{11} \geq 0.
\]  \hspace{1cm} (22)

(22) is seen in [NS].
As to Hadamard product, we have from the inequalities in the last section

\[
[(A \circ B)(\alpha)]^{-1} \leq (A \circ B)^{-1}(\alpha)
\]
\[
\leq A^{-1}(\alpha) \circ [B(\alpha)]^{-1}
\]
\[
\leq [(A^{-1} \circ B^{-1})^{-1}(\alpha)]^{-1}
\]
\[
\leq A^{-1}(\alpha) \circ B^{-1}(\alpha)
\]

and by (10)

\[
[(A \circ B)(\alpha)]^{-1} \leq [A(\alpha)]^{-1} \circ [B(\alpha)]^{-1}.
\]

But the right hand sides of (23) and (24), or equivalently \([(A \circ B)/\alpha]^{-1}
and \(A^{-1}/\alpha \circ B^{-1}/\alpha\), are not comparable, by the same example in Remark 2.2.

In addition, it is easy to prove, by (6), that

\[
A_{11} + B_{11}^{-1} \leq A_{11}^{-1} + B_{11}^{-1}.
\]

**Remark 3.1**  It is readily seen that

\[
A \leq B \Rightarrow A/\alpha \leq B/\alpha
\]

from the implications

\[
A \leq B \Rightarrow B^{-1} \leq A^{-1} \Rightarrow B_{11}^{-1} \leq A_{11}^{-1} \Rightarrow A_{11} \leq B_{11}.
\]

Note that the converse of the assertion is not true in general.

**Remark 3.2**  Main results of this paper can be easily generalized to more than two matrices.

**References**


