Angles, triangle inequalities, correlation matrices and metric-preserving and subadditive functions

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ABSTRACT

We present inequalities concerning the entries of correlation matrices, density matrices, and partial isometries through the positivity of $3 \times 3$ matrices. We extend our discussions to the inequalities concerning the triangle triplets with metric-preserving and subadditive functions.

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1. Introduction

We begin our discussions with angles between vectors by looking at two angle definitions and their characteristics. We first study the triangle inequalities for the angles through the positivity (i.e., positive semidefiniteness) of a $3 \times 3$ matrix; we investigate the relations between the triangle inequalities (of angles or more generally the triangle triplets) and metric-preserving and subadditive functions. Our results will capture some existing ones but using a different approach. The discussion on the $3 \times 3$ positive semidefinite matrices leads to some inequalities concerning the entries of correlation matrices with which we obtain inequalities for density matrices and partial isometries.

Let $V$ be an inner product space with the inner product $\langle \cdot , \cdot \rangle$ over the real number field $\mathbb{R}$. For any two nonzero vectors $u, v$ in $V$, there are two common ways to define the angle between the vectors $u$ and $v$ in terms of the inner product (see, for instance, [10, p. 58] and [11, p. 335], respectively):

$$\theta(u, v) = \arccos \frac{|\langle u, v \rangle|}{\|u\|\|v\|} \quad (1)$$

$$\Theta(u, v) = \arccos \frac{\langle u, v \rangle}{\|u\|\|v\|} \quad (2)$$

There are various reasons that the angles are defined in ways (1) and (2) (in the sense of Euclidean geometry). Definition (1) may stem from the angles between subspaces, while (2) makes perfect sense intuitively. We are interested in the properties of the angles regardless of their definitions.

Angle and inner product can be viewed as counterparts in a vector space. A vector may take a simple and familiar form like the ones in $\mathbb{R}^2$; it may look much more complicated like the elements (linear combinations of wedge products) in the Grassmann spaces [14, p. 172]. Some matrix functions are closely related to vectors and some types of products of vectors. It is a well-known fact that the trace of a matrix product is an inner product: $\text{tr}(AB) = \langle B, A^* \rangle$. The determinant (even more generally, the generalized matrix functions) can be expressed as an inner product of $*$-tensors (see, e.g., [14, p. 226]).

We will also need the term correlation matrix, which is a positive semidefinite matrix with all main diagonal entries equal to 1. Every positive semidefinite matrix with nonzero main diagonal entries can be normalized to a correlation matrix through scaling. The correlation matrices are frequently used in statistics. For its determinant and permanent properties, see, e.g., [16, 20, 23]. Our theorems rely on the results for the $3 \times 3$ correlation matrices.

In Section 2, we focus on the triangle inequalities through the positivity of $3 \times 3$ matrices. Our results provide a unified proof for the triangle inequalities for the angles $\theta$ and $\Theta$. We also present some relationships between the elements of correlation matrices. As applications, we obtain inequalities for density matrices and partial isometries. In
Section 3, we study the triangle triplets (which are more general than the angles formed by three vectors), metric-preserving and subadditive functions. Some inequalities of unit vectors are immediate from our results.

2. The triangle inequality and \(3 \times 3\) matrices

We extend somewhat the underlying number field of the vector space to the complex number field \( \mathbb{C} \) (see [3, p. 9]) and replace the angle in (2) by

\[
\Theta(u, v) = \arccos \frac{\Re(u, v)}{\|u\|\|v\|}
\]

(3)

The angles \(\theta\) and \(\Theta\) are closely related, but not equal unless \(\langle u, v \rangle\) is nonnegative. Since \(\Re(u, v) \leq |\langle u, v \rangle|\) and \(f(t) = \arccos t\) is a decreasing function in \(t \in [-1, 1]\), we have \(\Theta \geq \theta\). On the other hand, if \(\langle v, u \rangle \neq 0\), by taking \(p = \frac{(v, w)}{|(v, w)|}\), we get \(\theta(u, v) = \Theta(pu, v)\). It is easy to verify that (see, e.g., [13])

\[
\theta(u, v) = \min_{|p|=1} \Theta(pu, v) = \min_{|q|=1} \Theta(u, vq) = \min_{|p|=|q|=1} \Theta(pu, vq)
\]

(4)

For any nonzero vectors \(u, v \in V\), we see \(\theta(u, v) \in [0, \frac{\pi}{2}]\), \(\Theta(u, v) \in [0, \pi]\), and \(\theta(u, -u) = 0\), while \(\Theta(u, -u) = \pi\). The angle \(\theta\) defined in (1) between \(u\) and \(v\) is \(\frac{\pi}{2}\) if and only if \(u\) and \(v\) are orthogonal, i.e., \(\langle u, v \rangle = 0\); however, the angles \(\theta\) in general do not obey the law of cosines (for the triangle formed by nonzero vectors \(u, v\) and \(u - v\)). In contrast, the law of cosines does hold for the angles \(\Theta\), and \(\Theta(u, v) = \frac{\pi}{2}\) if and only if \(\langle u, v \rangle = 0\) over \(\mathbb{R}\), but it is possible for some vectors \(u, v\) to have an angle \(\Theta(u, v) = \frac{\pi}{2}\) and \(\langle u, v \rangle \neq 0\) over \(\mathbb{C}\).

The triangle inequalities for \(\theta\) and \(\Theta\) are known. That is, for all nonzero vectors \(u, v, w \in V\) and the angles \(\theta\) in (1) and \(\Theta\) in (3),

(t) \(\theta(u, v) \leq \theta(u, w) + \theta(w, v)\).

(T) \(\Theta(u, v) \leq \Theta(u, w) + \Theta(w, v)\).

The triangle inequality (T) is attributed to Krein [8] by Gustafson and Rao [7, p. 56]. The inequality was stated without proof in [8] and proved first in [17], then in [7, p. 56]. Note that the real case for (T) is also seen in [21, p. 31]. It has been observed [13] that (t) follows from (T) because of (4).

The proof of (T) in [7, p. 56] boils down to the positivity of the matrix

\[
R_Q = \begin{pmatrix}
1 & \Re(u, v) & \Re(u, w) \\
\Re(v, u) & 1 & \Re(v, w) \\
\Re(w, u) & \Re(w, v) & 1
\end{pmatrix}
\]

for unit vectors \(u, v, w\), which is ensured by the positivity of the Gram matrix.
\[ G_0 = \begin{pmatrix} 1 & \langle u, v \rangle & \langle u, w \rangle \\ \langle v, u \rangle & 1 & \langle v, w \rangle \\ \langle w, u \rangle & \langle w, v \rangle & 1 \end{pmatrix} \]

The positivity of \( G_0 \) also guarantees (see, e.g., [1, p. 26]) the positivity of

\[ A_0 = \begin{pmatrix} 1 & |\langle u, v \rangle| & |\langle u, w \rangle| \\ |\langle v, u \rangle| & 1 & |\langle v, w \rangle| \\ |\langle w, u \rangle| & |\langle w, v \rangle| & 1 \end{pmatrix} \]

which results in

\[ 1 + 2|\langle u, v \rangle||\langle v, w \rangle||\langle w, u \rangle| \geq |\langle u, v \rangle|^2 + |\langle v, w \rangle|^2 + |\langle w, u \rangle|^2 \] (5)

Inequality (5) is weaker than the following existing inequality (see (5.2) of Theorem 5.1 in [22]):

\[ 1 + 2\Re \langle u, v \rangle \langle v, w \rangle \langle w, u \rangle \geq |\langle u, v \rangle|^2 + |\langle v, w \rangle|^2 + |\langle w, u \rangle|^2 \] (6)

Using the idea of Gustafson and Rao we present a unified proof for the triangle inequalities (t) and (T) through 3 \times 3 matrices. We have known that 3 \times 3 matrices play important roles in geometry and analysis. For instance, the area of a triangle and the volume of a parallelepiped in \( \mathbb{R}^3 \), as well as the convexity of real-valued functions, can be computed and determined by (or through the determinants of) 3 \times 3 matrices (see, e.g., [15, p. 2]).

**Proposition 1.** Let \( a, b, c \) be real numbers such that the 3 \times 3 matrix

\[ B = \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix} \]

is positive semidefinite. Let \( f(t) \) be a strictly decreasing function defined on an interval \( \mathbb{I} \subseteq [0, \infty) \) with range \([-1, 1]\). If

\[ f(p + q) = f(p)f(q) - \sqrt{1 - f^2(p)}\sqrt{1 - f^2(q)} \] (7)

for all \( p, q, \) and \( p + q \) in \( \mathbb{I} \), then for \( x, y, z \) of any arrangement of \( a, b, c \),

\[ f^{-1}(x) \leq f^{-1}(y) + f^{-1}(z) \]

**Proof.** Since \( B \) is positive semidefinite, we have \(-1 \leq x, y, z \leq 1\) and

\[ 1 + 2xyz \geq x^2 + y^2 + z^2 \]
in which $x, y, z$ is any arrangement of $a, b, c$. The above inequality implies

$$(1 - y^2)(1 - z^2) \geq x^2 - 2xyz + (yz)^2 = (x - yz)^2$$

and

$$\sqrt{1 - y^2}\sqrt{1 - z^2} \geq |x - yz|$$

Hence

$$x \geq yz - \sqrt{1 - y^2}\sqrt{1 - z^2}$$

Note that as $f$ is decreasing, so is $f^{-1}$. If $f^{-1}(-1) \leq f^{-1}(y) + f^{-1}(z)$, then $f^{-1}(x) \leq f^{-1}(-1) \leq f^{-1}(y) + f^{-1}(z)$. Otherwise, $f^{-1}(-1) > f^{-1}(y) + f^{-1}(z) \geq f^{-1}(1) + 0 = f^{-1}(1)$; so, $f^{-1}(y) + f^{-1}(z)$ lies in the domain of $f$. Applying (7), we rewrite (8) as

$$f(f^{-1}(x)) \geq f(f^{-1}(y) + f^{-1}(z))$$

Since $f(x)$ is decreasing, we have $f^{-1}(x) \leq f^{-1}(y) + f^{-1}(z)$, as desired. \qed

One may verify that every function $f_r(t) = \cos(rt)$ with $r > 0$ on $[0, \pi/r]$ satisfies the condition (7).

**Corollary 2.** The following statements follow from Proposition 1 immediately.

(i) For any nonzero vectors $u, v,$ and $w$, inequalities (t) and (T) hold.

(ii) Let $0 \leq a, b, c \leq 1$ and denote $\alpha = \arccos a$, $\beta = \arccos b$, $\gamma = \arccos c$. Then all of the three inequalities $\alpha \leq \beta + \gamma$, $\beta \leq \gamma + \alpha$, and $\gamma \leq \alpha + \beta$ hold if and only if the matrix $B$ in Proposition 1 is positive semidefinite.

**Proof.** (i) In Proposition 1, setting $f(t) = \cos t$ on $[0, \pi]$ and taking the $3 \times 3$ matrix $B$ to be the matrices $A_0$ and $R_0$, we obtain the triangle inequalities (t) and (T), respectively. For (ii), one direction is clear. For the other direction, observe that $\alpha \leq \beta + \gamma \leq \pi$ when $0 \leq a, b, c \leq 1$ and that $0 \leq \beta - \gamma \leq \alpha$ or $0 \leq \gamma - \beta \leq \alpha$. Because cosine is decreasing on $[0, \pi]$, we have $bc - \sqrt{1 - b^2}\sqrt{1 - c^2} \leq a \leq bc + \sqrt{1 - b^2}\sqrt{1 - c^2}$. It follows that $1 + 2abc \geq a^2 + b^2 + c^2$. Hence, matrix $B$ is positive semidefinite. \qed

Note that for $(a, b, c) = (-1, -1, -1)$, we have $(\arccos a, \arccos b, \arccos c) = (\pi, \pi, \pi)$, the matrix $B$ in Proposition 1 is not positive semidefinite. Therefore, it is necessary to assume that $a, b, c$ are nonnegative in Corollary 2(ii). Note that Corollary 2(ii) has appeared in a recent paper [5, Proposition 1.4] with a different proof.

We now present some inequalities concerning the entries of correlation matrices. Although the following theorem is stated for the matrices of size $n \times n$, one can see that in essence it is really about $3 \times 3$ matrices.
Theorem 3. Let $A = (a_{ij})$ be an $n \times n$ complex correlation matrix. Then for all integers $1 \leq i, j, p, q \leq n$ such that $i < p < q$, $i < j < q$, and any real $k \geq 2$,

$$||a_{ip}||^k - |a_{iq}|^k| \leq \sqrt{1 - |a_{jq}|^{2k}} \leq \sqrt{1 - |a_{jq}|^{2k}}$$

**Proof.** This is immediate from Theorem 6 and Corollary 7 below. □

Note that every principal submatrix of a positive semidefinite matrix is again positive semidefinite and that if $M = (m_{ij})$ is a $3 \times 3$ positive semidefinite matrix, then so is $N = (|m_{ij}|)$. (This is not true for $4 \times 4$ or higher dimensions.)

Density matrices play an important role in quantum computation; density matrices are the positive semidefinite matrices having trace 1.

Corollary 4. Let $G_i$ be density matrices and let $G_i = H_i^* H_i$, where $H_i$ are $n \times n$ matrices, $i = 1, 2, \ldots, n$. Then for all $i < p < q$, $i < j < q$, and real $k \geq 2$,

$$||\text{tr} H_i^* H_p| - |\text{tr} H_i^* H_q|| \leq \sqrt{1 - |\text{tr} H_j^* H_q|^2} \leq \sqrt{1 - |\text{tr} H_j^* H_q|^2}$$

**Proof.** If the partitioned matrix $A = (A_{ij})$ is positive semidefinite, then so is $(\text{tr} A_{ij})$ (see, e.g., [22]). Note that the matrix $(\text{tr} H_i^* H_j)$ is correlation. The assertion for $|H_i^* H_j|$ is due to a recent result of Drury [5] (see also [12]). □

Corollary 5. Let $H_1, H_2, \ldots, H_n$ be partial isometries each having $n$ columns, i.e., every $H_i^* H_i = I_n$. Then for all $i < p < q$, $i < j < q$, and real $k \geq 2$,

$$||\det H_i^* H_p| - |\det H_i^* H_q|| \leq \sqrt{1 - |\det H_j^* H_q|^2} \leq \sqrt{1 - |\det H_j^* H_q|^2}$$

**Proof.** If the partitioned matrix $A = (A_{ij})$ is positive semidefinite, then so is $(\det A_{ij})$ (see, e.g., [22]). Note that the matrix $(\det H_i^* H_j)$ is correlation. □
The proof of Theorem 3 reduces to the following results on $3 \times 3$ correlation matrices. These results can be stated in terms of complex matrices for which $a, b, c$ in the inequalities are replaced by $\text{Re } a, \text{Re } b, \text{Re } c$ or $|a|, |b|, |c|$, respectively.

**Theorem 6.** Let $a, b, c$ be real numbers such that the $3 \times 3$ matrix

$$B = \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix}$$

is positive semidefinite. Denote

$$c_- = ab - \sqrt{(1-a^2)(1-b^2)}, \quad c_+ = ab + \sqrt{(1-a^2)(1-b^2)}$$

and let

$$\Delta_{a,b} = \max\{\sqrt{1-c_-^2}, \sqrt{1-c_+^2}\}, \quad \delta_{a,b} = \min\{\sqrt{1-c_-^2}, \sqrt{1-c_+^2}\}$$

(i) If $f(x)$ is a function defined on the interval $[c_-, c_+]$ such that

$$\delta_{a,b} \leq f(x), \quad \text{for all } x \in [c_-, c_+]$$

then

$$|a^2 - b^2| \leq \Delta_{a,b} f(x), \quad \text{for all } x \in [c_-, c_+] \quad (9)$$

In particular

$$|a^2 - b^2| \leq \Delta_{a,b} \sqrt{1-c^2} \quad (10)$$

(ii) If $g(x)$ is a function defined on the interval $[c_-, c_+]$ such that

$$\sqrt{1-c^2} \leq g(x), \quad \text{for all } x \in [c_-, c_+]$$

then

$$|a - b| \leq \sqrt{1-c_-} \cdot g(x), \quad \text{for all } x \in [c_-, c_+] \quad (11)$$

In particular

$$|a - b| \leq \sqrt{1-c_-} \cdot \sqrt{1-c} \quad (12)$$

(iii) If $a, b, c$ are in $[0, 1]$, then

$$|a - b| \leq \sqrt{1-c^2} \quad (13)$$
Proof. (i) Observe that $B$ is positive semidefinite if and only if $a, b, c \in [-1, 1]$ and $1 + 2abc \geq a^2 + b^2 + c^2$ and that $1 + 2abc \geq a^2 + b^2 + c^2$ if and only if $m(c) := c^2 - 2abc - (1 - a^2 - b^2) \leq 0$. Note that $m(c) = 0$ has solutions $c_-, c_+$. We see that $m(x) \leq 0$ for all $x \in [c_-, c_+]$. Moreover, since $B$ is positive semidefinite, the scalar $c$ in the matrix $B$ lies in $[c_-, c_+]$.

Let $a = \cos \mu, b = \cos \nu, \mu, \nu \in [0, \pi]$. Then $c_{\pm} = \cos(\mu \mp \nu)$. We compute

$$|a^2 - b^2| = |\cos^2 \mu - \cos^2 \nu| = |\sin(\mu + \nu) \sin(\mu - \nu)|$$

$$= \sqrt{1 - c_{\pm}^2} \cdot \sqrt{1 - c_{\mp}^2}$$

$$= \max\{\sqrt{1 - c_{\pm}^2}, \sqrt{1 - c_{\mp}^2}\} \cdot \min\{\sqrt{1 - c_{\pm}^2}, \sqrt{1 - c_{\mp}^2}\}$$

$$\leq \Delta_{a,b} f(x)$$

which is inequality (9). Taking $f(x) = \sqrt{1 - x^2}$, we have $f(x) \geq \delta_{a,b}$ for all $x \in [c_-, c_+]$. This leads to (10) by setting $x = c$.

(ii) In a similar way by using trigonometric identities, we have

$$|a - b| = |\cos \mu - \cos \nu| = \sqrt{1 - \cos(\mu + \nu)} \cdot \sqrt{1 - \cos(\mu - \nu)}$$

$$= \sqrt{1 - c_{\pm}} \cdot \sqrt{1 - c_{\mp}}$$

$$\leq \sqrt{1 - c_{\pm}} \cdot g(x)$$

The special case (12) is because $\sqrt{1 - c_{\mp}} \leq \sqrt{1 - c}$ for $c \in [c_-, c_+]$.

(iii) If $0 \leq a, b, c \leq 1$, then $1 + 2abc \geq a^2 + b^2 + c^2$ yields

$$|a - b|^2 \leq -c^2 + 2abc + 1 - 2ab = 1 - c^2 + 2ab(c - 1) \leq 1 - c^2$$

Thus inequality (13) follows. 

Note that $y = 1 + \frac{1}{c_{\pm}}(\sqrt{1 - c_{\pm}^2} - 1)x$ with $c_{\pm} \neq 0$ is also a function satisfying the conditions (i) and (ii) in Theorem 6. Equalities in (10) and (12) occur when $B$ is the positive semidefinite matrix with $a = 1, b = c = 0$. So in this sense the upper bounds for these inequalities are optimal.

We point out that the restriction on $a, b, c$ in (13) being nonnegative cannot be removed. For instance, take $a = 1, b = -1$ and $c = -1$. Then $B$ is positive semidefinite. However, $|a - b| > \sqrt{1 - c^2}$. Moreover, inequality (10) reveals

$$|a^2 - b^2| \leq \sqrt{1 - c^2}$$

as a sister inequality of (13). For $k \geq 3, |a^k - b^k|$ is not bounded by $\sqrt{1 - c^2}$ in general. One may verify by the following example that $|a^3 - b^3| > \sqrt{1 - c^2}$. Let
\[
C = \begin{pmatrix}
1 & 1 & 0.1 \\
1 & 1 & 0.1 \\
0.1 & 0.1 & 1
\end{pmatrix}
\]

Then \( C \) is positive semidefinite. For \( a = 1, b = c = 0.1 \), we have

\[
|a - b| = 0.9, \quad |a^2 - b^2| = 0.99, \quad |a^3 - b^3| = 0.999, \quad \sqrt{1 - c^2} \approx 0.995
\]

For \( x, y, z \) of any arrangement of \( a, b, c \) in the matrix \( B \), Theorem 6 gives

\[
|x^2 - y^2| \leq \sqrt{1 - z^2}
\]

and

\[
|x - y| \leq \sqrt{2} \cdot \sqrt{1 - z}
\]

Inequalities (13) and (14) imply that for any unit vectors \( u, v, w \) and \( k \geq 1, 2 \),

\[
||\langle u, v \rangle|^k - |\langle u, w \rangle|^k| \leq \sqrt{1 - |\langle w, v \rangle|^2}
\]

Note that the positivity of \( B \) in the previous theorem is equivalent to \( 1 + 2abc \geq a^2 + b^2 + c^2 \) for real numbers \( a, b, c \) in \([-1, 1]\), in which \( a, b, c \) are symmetric.

**Corollary 7.** Let \( a, b, c \) be real numbers such that the \( 3 \times 3 \) matrix

\[
B = \begin{pmatrix}
1 & a & b \\
a & 1 & c \\
b & c & 1
\end{pmatrix}
\]

is positive semidefinite. Then the following statements hold.

(i) \( ||a| - |b|| \leq \sqrt{1 - |c|^2} \leq \sqrt{2} \cdot \sqrt{1 - |c|} \) and for real \( k \geq 2 \),

\[
|a|^k - |b|^k| \leq \sqrt{1 - |c|^k}
\]

(ii) If \( 0 \leq \alpha, \beta, \gamma \leq \pi/2 \) and \( |\alpha - \beta| \leq \gamma \leq \alpha + \beta \), then for any integer \( k \geq 1 \)

\[
|\cos^k \alpha - \cos^k \beta| \leq \sqrt{k} \cdot \sin \gamma
\]

**Proof.** (i) Recall that if \( M = (m_{ij}) \) is a \( 3 \times 3 \) positive semidefinite matrix, then \( (|m_{ij}|) \) is also positive semidefinite. The first inequality in (i) is immediate from (13). From a result of FitzGerald and Horn \([6, \text{Theorem 2.2}]\), we know that if \( P = (p_{ij}) \) is \( 3 \times 3 \) nonnegative positive semidefinite, then the Hadamard power matrix \( (p_{ij}^k) \) is positive semidefinite for
all real $r \in [1, \infty)$. Thus for the $3 \times 3$ positive semidefinite matrix $B = (b_{ij})$ and for any real $k \geq 2$, matrix $(|b_{ij}|^{k/2})$ is positive semidefinite. An application of (14) implies

$$||a|^k - |b|^k| = \left|\left(|a|^{k/2}\right)^2 - \left(|b|^{k/2}\right)^2\right| \leq \sqrt{1 - \left(|c|^{k/2}\right)^2} = \sqrt{1 - |c|^k}$$

(ii) Let $a = \cos \alpha, b = \cos \beta, c = \cos \gamma$. By Corollary 2(ii), the matrix $B$ is positive semidefinite; so is the Hadamard power matrix $(b_{ij}^k)$. For $k = 1$, $|a - b| \leq \sqrt{1 - c^2}$ is the same as $|\cos \alpha - \cos \beta| \leq \sin \gamma$. For integer $k > 1$,

$$|a^k - b^k| \leq \sqrt{1 - c^{2k}} = \sqrt{1 - c^2} \cdot \sqrt{1 + c^2 + \cdots + c^{2(k-1)}} \leq \sqrt{k} \cdot \sqrt{1 - c^2}$$

which is the same as the desired inequality. \Box

In the following correlation matrix $D$, $\sqrt{1 - c} = 0.3 < |a - b| = 0.4$. This shows that the $\sqrt{2}$ in Corollary 7(i) cannot be replaced by 1 in general.

$$D = \begin{pmatrix} 1 & 0 & 0.4 \\ 0 & 1 & 0.91 \\ 0.4 & 0.91 & 1 \end{pmatrix}$$

In the previous proof, we saw $|\cos \alpha - \cos \beta| \leq \sin \gamma$. Inequality (14) implies $|\cos^2 \alpha - \cos^2 \beta| \leq \sin \gamma$. A question arises: Can the $\sqrt{k}$ in Corollary 7(ii) be removed or replaced by a constant (like $\sqrt{2}$) that is independent of $k$? The following result gives a negative answer.

**Theorem 8.** Let $0 \leq \alpha, \beta, \gamma \leq \frac{\pi}{2}$, $|\alpha - \beta| \leq \gamma \leq \alpha + \beta$ and let

$$R_k = R_k(\alpha, \beta, \gamma) = \left|\frac{\cos^k \alpha - \cos^k \beta}{\sin \gamma}\right|, \quad \gamma \neq 0$$

Then for sufficiently large positive integer $k$,

$$\sup_{\alpha, \beta, \gamma} R_k \approx \sqrt{\frac{k}{e}}$$

**Proof.** If $\alpha = \beta$ then $R_k = 0$. In addition, $R_k$ is symmetric with respect to $\alpha$ and $\beta$. So, for the maximum $R_k$, we may assume $\alpha > \beta$. Moreover,

$$\frac{\cos^k \beta - \cos^k \alpha}{\sin \gamma} \leq \frac{\cos^k \beta - \cos^k \alpha}{\sin(\alpha - \beta)}$$

Let $R_k(\alpha, \beta) = \frac{\cos^k \beta - \cos^k \alpha}{\sin(\alpha - \beta)}$ if $\alpha \neq \beta$ and $R_k(\alpha, \beta) = k \sin \beta \cos^{k-1} \beta$ if $\alpha = \beta$ (denoted by $R_k(\beta)$ for short). Consider $R_k(\alpha, \beta)$ over the triangular region $\Delta = \{(\alpha, \beta) \mid \frac{\pi}{2} \geq \alpha \geq \beta \geq 0\}$

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\( \beta \geq 0 \). For every given integer \( k \geq 2 \), the function \( R_k(\alpha, \beta) \) is continuous on \( \mathbb{D} \). To see this, it is sufficient to notice that

\[
\lim_{(\alpha, \beta) \to (t,t)} R_k(\alpha, \beta) = \lim_{(\alpha, \beta) \to (t,t)} \frac{(\cos \beta - \cos \alpha)(\sum_{l=0}^{k-1} \cos^{k-1-l} \beta \cos^l \alpha)}{\sin(\alpha - \beta)}
\]

\[
= \lim_{(\alpha, \beta) \to (t,t)} \frac{\sin(\frac{\beta + \alpha}{2})(\sum_{l=0}^{k-1} \cos^{k-1-l} \beta \cos^l \alpha)}{\cos(\frac{\alpha - \beta}{2})}
\]

\[
= k \sin t \cos^{k-1} t = R_k(t), \quad \text{where } t \in \left[ 0, \frac{\pi}{2} \right].
\]

Thus \( R_k(\alpha, \beta) \) attains its maximum value at some point(s) in \( \mathbb{D} \). On the \( \alpha = \beta \) portion of the boundary of \( \mathbb{D} \), by computing the critical number, the function \( R_k(t) = k \sin t \cos^{k-1} t \) is maximized when \( \tan t = \frac{1}{\sqrt{k-1}} \), and we get

\[
\sup_{t \in [0, \frac{\pi}{2}]} R_k(t) = \frac{k}{\sqrt{k-1}} \left( 1 - \frac{1}{k} \right)^{k/2} \quad \text{(which is unbounded as } k \to \infty \text{)}
\]

For any given (small) \( \varepsilon > 0 \), if \( \alpha - \beta \geq \varepsilon \), then \( R_k(\alpha, \beta) \leq \frac{1}{\sin \varepsilon} \) uniformly for all \( k \). So, in view of the behavior of \( R_k(\alpha, \beta) \) for \( \alpha = \beta \) in the above discussion, for all \( k \) large enough, \( R_k(\alpha, \beta) \) is maximized as \( \alpha \) approaches \( \beta \). That is, if we let \( \alpha = \beta + \varepsilon, \varepsilon > 0 \), then

\[
\sup_{\alpha, \beta, \gamma} R_k(\alpha, \beta) = \sup_{\alpha, \beta} R_k(\alpha, \beta) = \sup_{\beta} \lim_{\varepsilon \to 0} R_k(\beta + \varepsilon, \beta)
\]

\[
= \sup_{\beta} R_k(\beta) = \frac{k}{\sqrt{k-1}} \left( 1 - \frac{1}{k} \right)^{k/2}
\]

\[
\approx \sqrt{\frac{k}{\varepsilon}} \quad \text{for sufficiently large } k \quad \square
\]

**Remark.** In fact, one may show through some routine calculus computation that for each \( R_k(\alpha, \beta) \) with \( k \geq 1 \), there are no interior critical points in \( \mathbb{D} \).

### 3. Metric-preserving and subadditive functions

Let \( a, b, c \) be nonnegative numbers. The triplet \((a, b, c) \in \mathbb{R}^3\) is said to be triangle if \( a \leq b + c \), \( b \leq a + c \), and \( c \leq a + b \); equivalently, \( |a - b| \leq c \leq a + b \). This can be restated as \( x \leq y + z \), where \( x, y, z \) is any arrangement of \( a, b, c \); equivalently, \( |x - y| \leq z \) for any arrangement of \( a, b, c \). From the inequalities (t) and (T) of Section 2, we see for any nonzero vectors \( u, v, w \) in an inner product space, \((\theta(u, v), \theta(v, w), \theta(w, u))\) and \((\Theta(u, v), \Theta(v, w), \Theta(w, u))\) are triangle triplets. They are special members of the set of all triangle triplets:
\[ \Delta = \{ (a, b, c) \mid a, b, c \geq 0, \ |a - b| \leq c \leq a + b \} \]

For \( a, b, c \) in \([0, 1]\), by Corollary 2(ii), the triplet \((\arccos a, \arccos b, \arccos c)\) is triangle if and only if the \(3 \times 3\) matrix \(B\) in Proposition 1 is positive semidefinite. Proposition 1 reveals a relation between the positivity of the \(3 \times 3\) matrix \(B\) and triangle triplets via certain functions. This section presents a theorem of this type, with which we show some inequalities for unit vectors.

Let \( f \) be a nonnegative function defined on \([0, \infty)\). We say that \( f \) is metric-preserving (a metric preserver) if \( f \circ d \) is also a metric on \( M \), where \((M, d)\) is any metric space, triangle-preserving (a triangle preserver) if \((f(a), f(b), f(c))\) is triangle whenever \((a, b, c)\) is triangle, and subadditive if \(f(s + t) \leq f(s) + f(t)\) for all \(s, t \geq 0\). The reader is referred to [2] for metric-preserving functions and [9, Chapter 16] and [18, Chapter 12] for subadditive functions.

These three functions are closely related, but not exactly the same. It is known (see, e.g., [4, p. 9]) that if \(f : [0, \infty) \mapsto [0, \infty)\) is nondecreasing and subadditive then \(f\) is a triangle preserver. A nonnegative concave function vanishing at 0 is necessarily subadditive (see, e.g., [18, p. 314]). Nonnegative concave (not necessarily continuous) functions must be nondecreasing. (This seems to be a known fact; but we were not able to find a reference with a proof.) A stronger version of the result is stated as:

**Proposition 9.** Let \(L = [l, \infty)\) or \((l, \infty) \subseteq \mathbb{R}\). If \(f\) is nonnegative on \(L\), i.e., \(f(x) \geq 0\) for all \(x \in L\), and if \(f\) is mid-point concave on \(L\), i.e.,

\[ f\left(\frac{1}{2}x + \frac{1}{2}y\right) \geq \frac{1}{2}f(x) + \frac{1}{2}f(y), \quad \text{for all } x, y \in L \]

then \(f(x)\) is monotonically increasing on \(L\), i.e., \(f(x) \geq f(y)\) for \(x > y, x, y \in L\).

**Proof.** Suppose that \(f(x)\) is not monotonically increasing. Then there exist \(s, t \in L\), \(t > 0\), such that \(f(s) > f(s + t)\). Let \( r = f(s) - f(s + t) > 0 \). Since

\[ f(s + t) = f\left(\frac{1}{2}s + \frac{1}{2}(s + 2t)\right) \geq \frac{1}{2}f(s) + \frac{1}{2}f(s + 2t) \]

we arrive at

\[ f(s + t) - f(s + 2t) \geq f(s) - f(s + t) = r \]

Let

\[ F_n = f(s + nt) - f(s + (n + 1)t), \quad n = 0, 1, 2, \ldots \]

Then, in a similar way as above, we can show that \(\{F_n\}\) is a decreasing sequence bounded by \(r\) from below, i.e., \(F_n \geq F_{n-1} \geq \cdots \geq F_1 \geq r\). It follows that
\[ f(s) - f(s + nt) = F_0 + F_1 + F_2 + \cdots + F_{n-1} \geq nr. \]

Thus, \( f(s + nt) < 0 \) when \( n \) is large enough, contradicting \( f(x) \geq 0, \ x \in L. \)

**Theorem 10.** Let \( a, b, c \) be real numbers such that the 3 \( \times \) 3 matrix

\[
B = \begin{pmatrix}
1 & a & b \\
a & 1 & c \\
b & c & 1
\end{pmatrix}
\]

is positive semidefinite. Then for all functions \( f \) described in Proposition 1 and for all metric-preserving functions \( g, \) with \( h = g \circ f^{-1}, \) we have

\[
h(a) \leq h(b) + h(c)
\]

Consequently, for any real \( k \geq 2, \)

\[
k^{\frac{1}{k}} \sqrt{1 - |a|^k} \leq k^{\frac{1}{k}} \sqrt{1 - |b|^k} + k^{\frac{1}{k}} \sqrt{1 - |c|^k}
\]

**Proof.** Proposition 1 says that \((f^{-1}(a), f^{-1}(b), f^{-1}(c))\) is a triangle triplet. For any metric-preserving function \( g, \) with \( h = g \circ f^{-1}, \) \((h(a), h(b), h(c))\) is also a triangle triplet. This gives the desired inequality.

For the second part, for any fixed real \( k \geq 2, \) consider the function

\[
p(t) = \begin{cases}
\sqrt[k]{1 - \cos^k t} & \text{if } 0 \leq t \leq \frac{\pi}{2} \\
1 & \text{if } t > \frac{\pi}{2}
\end{cases}
\]

It is straightforward to verify that \( p(t) \) is nonnegative, increasing, and concave for \( k \geq 2 \) (by checking \( p'(t) \geq 0, \ p''(t) \leq 0); \) thus, \( p(t) \) is metric-preserving. Applying \( p(t) \) to \((\arccos |a|, \arccos |b|, \arccos |c|)\) yields the inequality. □

**Corollary 11.** Let \( u, v, w \) be any unit vectors of an inner product space. Then

\[
k^{\frac{1}{k}} \sqrt{1 - |\langle u, v \rangle|^k} \leq k^{\frac{1}{k}} \sqrt{1 - |\langle u, w \rangle|^k} + k^{\frac{1}{k}} \sqrt{1 - |\langle w, v \rangle|^k}
\]

(16)

for any real number \( k \geq 2. \) In particular,

\[
\sqrt{1 - |\langle u, v \rangle|^2} \leq \sqrt{1 - |\langle u, w \rangle|^2} + \sqrt{1 - |\langle w, v \rangle|^2}
\]

(17)

Similar inequalities hold for \( |\text{Re} \langle \cdot, \cdot \rangle|\) in place of \( |\langle \cdot, \cdot \rangle|\).

Inequality (17) appears in [19] (see also [21, p. 195]). Inequality (16) is seen in [13] (with a minor glitch on the condition \( b + c \leq 1 \) which can be fixed).
Corollary 12. Let $u, v, w$ be unit vectors in an inner product space, $\alpha, \beta, \gamma$ be respectively the angles $\theta(u,v), \theta(v,w), \theta(w,u)$ or $\Theta(u,v), \Theta(v,w), \Theta(w,u)$. Then

(i) $\alpha \leq \beta + \gamma$.
(ii) $\sin \alpha \leq \sin \beta + \sin \gamma$.
(iii) $\cos \alpha \leq \cos \beta + \cos \gamma$ in general.
(iv) $\cos \alpha \leq \cos \beta + \sin \gamma$.

Proof. (i) is the same as (t) and (T) in Section 2. (ii) is true because of (17) and the similar inequality for $\Re \langle \cdot, \cdot \rangle$. For (iii), take $u = (0, 0, 1)$, $v = \frac{1}{\sqrt{2}}(1, 0, 1)$, and $w = (0, 1, 0)$ in $\mathbb{R}^3$ with the standard Euclidean inner product. (iv) follows from inequality (13). □

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References