Eigenvalue Inequalities for Matrix Product
Fuzhen Zhang and Qingling Zhang

Abstract—We present a family of eigenvalue inequalities for the product of a Hermitian matrix and a positive–semidefinite matrix. Our theorem contains or extends some existing results on trace and eigenvalues.

Index Terms—Eigenvalue, inequality, singular value, trace.

I. INTRODUCTION

Because of deep connections with control theory, dynamical systems, and other related areas, Lyapunov and Riccati equations have been extensively studied, and the problem of estimating the solutions to the equations is of central importance. In [1], some lower and upper bounds for the trace of the solutions to both equations are explicitly presented, while in [2]–[9], much work is done particularly on the bounds of traces and the extreme eigenvalues of matrix product. We take, for instance, the Lyapunov equation in the matrix form

$$A^T X + X A = -Q$$ (LE)

where $Q$ is a real positive–definite matrix. The equation has a positive–definite solution $X$ if and only if matrix $A$ is stable, i.e., all of its eigenvalues have negative real parts. Given a stable matrix $A$ and a matrix $Q$ of the same size, by taking trace, a necessary condition for $X$ to be a solution to the equation is

$$\text{tr}(X A) = -\frac{1}{2} \text{tr}(Q).$$

This is one of the reasons leading to the study of the trace of the product of two matrices, one of which is assumed to be positive semidefinite.

It is shown in [1, Lemma 1] that for real symmetric matrix $A$ and real positive–semidefinite (or nonnegative definite) matrix $B$ of the same size

$$\lambda_{\min}(A) \text{tr}(B) \leq \text{tr}(AB) \leq \lambda_{\max}(A) \text{tr}(B)$$ (1)

where $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ are the smallest and largest eigenvalues of $A$, respectively. We note that (1) holds when $A$ and $B$ are switched in (1). With these inequalities, some trace bounds on the solutions of the algebraic matrix Riccati and Lyapunov equations are obtained in [1].

Under the condition that $A$ be arbitrary and $B$ be positive semidefinite, both $n \times n$ and real, it is proven in [2] that

$$\text{tr}(AB) \leq |\text{tr}(AB)| \leq |A|_2 \text{tr}(B)$$ (2)

where $|A|_2 = \sigma_{\max}(A)$ is the spectral norm, i.e., the largest singular value of $A$. In the same setting, with $\bar{A} = (1/2)(A + A^T)$, it is stated in [3, Th. 2] that

$$-\lambda_{\max}(-\bar{A}) \text{tr}(B) \leq \text{tr}(AB) \leq \lambda_{\max}(\bar{A}) \text{tr}(B).$$ (3)

The inequalities in (3) are extended in [5, Lemma II.1] to complex matrices with the eigenvalues of $A$ and $B$ involved in a symmetric form: For $n \times n$ Hermitian matrices $A$ and $B$, assuming that all eigenvalues are arranged in decreasing order $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$ for $n \times n$ Hermitian matrix $X$, then

$$\sum_{i=1}^{n} \lambda_{n-i+1}(A) \lambda_i(B) \leq \text{tr}(AB) \leq \sum_{i=1}^{n} \lambda_i(A) \lambda_i(B).$$ (4)

More generally, [5, Lemma II.2] for any complex $A$ and Hermitian $B$, both $n \times n$

$$\sum_{i=1}^{n} \lambda_{n-i+1}(\bar{A}) \lambda_i(B) \leq \text{tr}(AB) \leq \sum_{i=1}^{n} \lambda_i(\bar{A}) \lambda_i(B).$$ (5)

where $\bar{A} = (1/2)(A + A^*)$ and $A^* = (\bar{A})^T$ is the conjugate transpose of the matrix $A$. In particular, if $A$ and $B$ are $n \times n$ real and if, in addition, $B$ is symmetric, then

$$\sum_{i=1}^{n} \lambda_{n-i+1}(\bar{A}) \lambda_i(B) \leq \text{tr}(AB) \leq \sum_{i=1}^{n} \lambda_i(\bar{A}) \lambda_i(B).$$ (6)

Relaxing the symmetry of $B$ and using singular values (arranged in decreasing order), it is shown in [8, Th. 1] that for any $n \times n$ real matrices $A$ and $B$

$$\lambda_{\min}(\bar{A} S) \sum_{i=1}^{n} \sigma_i(B) \leq \text{tr}(AB) \leq \lambda_{\max}(\bar{A} S) \sum_{i=1}^{n} \sigma_i(B)$$ (7)

where $S = UV^T$ if $B = UDV^T$ is a singular value decomposition of $B$.

It seems that much of the effort has been made on removing the symmetry or the positive definiteness of $A$ and $B$, though the proofs of most of the results boil down to those of Hermitian or symmetric matrices. We note that for any square complex matrix $A$, if we denote by $\bar{A} = (1/2)(A + A^*)$ and $A^* = (1/2)(A + A^*)$, then $\text{tr}(AB) = \text{tr}(\bar{A} B)$ if $B$ is symmetric, and $\Re(\text{tr}(AB)) = \text{tr}(\bar{A} B)$ if $B$ is Hermitian. Thus for any $n \times n$ matrices $A$ and $B$, $\text{tr}(AB) = \text{tr}(\bar{A} B)$ and $\Re(\text{tr}(AB)) = \text{tr}(\bar{A} B)$. Obviously, $\bar{A} = A$ when $A$ is a real matrix.

We remark that (1) is immediate from (4) which is the same as our later (15) and that (7) [or our (20)] implies (3) which is stronger than (2) because $\lambda_{\max}(A) \leq ||A||_2$ (see, for instance, [13, p. 236]).

In view that the trace $\text{tr}(AB)$ is the sum of all eigenvalues of the matrix product $AB$, we will extend the sum to partial sum and arrive at a family of eigenvalue inequalities which yield the aforementioned inequalities. We explain why our results may be regarded as the most general ones in certain sense. At end we derive a variety of inequalities from our theorem and present an application.
II. MAIN RESULTS

Throughout this note, $I_n$ is the $n \times n$ identity matrix and the eigenvalues ($\lambda_i$), if all real, and the singular values ($\sigma_i$) are always arranged in decreasing order. Moreover we assume that all the matrices are complex and of size $n \times n$ unless otherwise stated. Our results are of course valid for real matrices.

We proceed with two basic known results in matrix theory as our lemmas. These results can be found in [10, pp. 511 and 513], respectively.

**Lemma 1:** If $H$ is an $n \times n$ Hermitian matrix, then for every $k$, $1 \leq k \leq n$

$$\sum_{i=1}^{k} \lambda_i(H) = \max_{U \in \mathbb{C}^{n \times k}} \text{tr}(UHU^*)$$

where the extrema are over all $k \times n$ complex matrices $U$ satisfying $UU^* = I_k$.

**Lemma 2:** If $H$ is an $n \times n$ Hermitian matrix and $V$ is a $k \times n$ matrix, $k \leq n$, then

$$\sum_{i=1}^{k} \lambda_i(H)\lambda_i(VV^*) \geq \text{tr}(VHV^*) = \sum_{i=1}^{k} \lambda_{n-i+1}(H)\lambda_i(VV^*).$$

Now, we present our main result.

**Theorem 3:** Let $H$ be an $n \times n$ Hermitian matrix and $P$ be an $n \times n$ positive–semidefinite matrix. Then, for every integer $k$, $1 \leq k \leq n$,

$$\max_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{i=1}^{k} \lambda_{n-i+1}(H)\lambda_{i}(P) \leq \sum_{i=1}^{k} \lambda_i(H)\lambda_i(P) \leq \max_{U \in \mathbb{C}^{n \times k}} \text{tr}(UPU^*).$$

Proof: First notice (see, for instance, [13, p. 51]) that $H \in H = H^{(1)/(2)}P^{(1)/(2)}$ and $P^{(1)/(2)}H^{(1)/(2)}$ have the same eigenvalues, where $P^{(1)/(2)}$ is the unique positive semidefinite square root matrix of $P$. By Lemma 1, for each $k$, where $1 \leq k \leq n$, we have

$$\sum_{i=1}^{k} \lambda_i(H)\lambda_i(P) \geq \max_{U \in \mathbb{C}^{n \times k}} \text{tr}(UPU^*) \geq \sum_{i=1}^{k} \lambda_{n-i+1}(H)\lambda_{i}(P).$$

Let $P^{(1)/(2)} = R^*DR$, where $R$ is a unitary matrix and $D$ is a diagonal matrix with $\lambda_{i_1}^{(1)/(2)}(P)$, $\lambda_{i_2}^{(1)/(2)}(P)$, $\ldots$, $\lambda_{i_k}^{(1)/(2)}(P)$ and the square roots of the rest eigenvalues of $P$ on the main diagonal. Set $U_0 = (I_k, 0 \times 0)$. Then

$$\max_{U \in \mathbb{C}^{n \times k}} \text{tr}(UPU^*) \geq \text{tr}(U_0P^{1/2}HP^{1/2}U_0^*) = \max_{i=1}^{k} \sum_{i=1}^{k} \lambda_{n-i+1}(H)\lambda_{i}(P).$$

To show the second inequality in (8), use Lemma 2 to

$$\text{tr}(UP^{(1)/(2)}H^{(1)/(2)}U^*) = \sum_{i=1}^{k} \lambda_i(H)\lambda_i(UPU^*).$$

The desired inequality then follows immediately. This completes the proof.

Note that the proof in [5] by substituting $H$ with a positive semidefinite $H + \alpha I$ (for sufficiently large $\alpha$) does not work here, since a partial sum is no longer a trace in general.

To compare with existing results, it would be appealing if the second inequality in (8) held without the appearance of $U$, as is well known ([13, p. 232] (see also [12] for more general inequalities), if $H$ and $P$ are both positive semidefinite (see [10, p. 242] for an analog for $H + P$), then for $k = 1, 2, \ldots, n$

$$\sum_{i=1}^{k} \lambda_i(H)\lambda_i(P) \leq \sum_{i=1}^{k} \lambda_i(H)\lambda_i(P).$$

This is not true when $H$ is Hermitian in general. A counterexample comes handy: Set $H \equiv -I_3$, take $P$ to be the diagonal matrix $\text{diag}(3, 2, 1)$ and $k = 1$. Moreover, recall [11] that when $H$ and $P$ are both positive semidefinite,

$$\sum_{i=1}^{k} \lambda_i(H)\lambda_i(P) \geq \sum_{i=1}^{k} \lambda_{n-i+1}(H)\lambda_{i}(P).$$

With this in mind we would hope that the summand in the first term in (8) could more generally be $\lambda_{n-i+1}(H)\lambda_{i}(P)$ in place of $\lambda_{n-i+1}(H)\lambda_{i}(P)$. This is not true in general. Take $P = \text{diag}(3, 2, 1)$ and $H = \text{diag}(-1, -2, -3)$ with $k = 2$ and $t_1 = 2, t_2 = 3$. Then, the right-hand side of (10) is $-5$, while the left-hand side is $-6$. So (9) and (10) no longer hold if one matrix is Hermitian, but not positive semidefinite. On the other hand, it is also known ([5] or [10, p. 262]) that when $H$ and $P$ are both Hermitian

$$\sum_{i=1}^{n} \lambda_i(H)\lambda_i(P) \leq \sum_{i=1}^{n} \lambda_i(H)\lambda_i(P).$$

This obviously follows from our theorem, since $\lambda_i(UPU^*) = \lambda_i(P)$ when $k = n$. Note that the upper index $n$ in (11) cannot be replaced by $k$ in general.

In light of inequalities (9)–(11) and in the sense of going from non-negative definiteness to Herimity, that is, one matrix is Hermitian and the other one is positive semidefinite, our theorem may be regarded as the most general one.

III. COROLLARIES

We now present variances of (8) and derive more inequalities from it, some of which may have existed possibly in different forms. Replacing $H$ with $-H$ in (8) and noting that $\lambda_i(-H) = -\lambda_{n-i+1}(H)$, we have

**Corollary 4:** Let $H$ be Hermitian and $P$ be positive semidefinite. Then

$$\sum_{i=1}^{k} \lambda_i(H)\lambda_i(P) \geq \sum_{i=1}^{k} \lambda_{n-i+1}(H)\lambda_i(P) \geq \min_{U \in \mathbb{C}^{n \times k}} \sum_{i=1}^{k} \lambda_{n-i+1}(H)\lambda_i(UPU^*).$$
Setting $t_1 = i$ in (8) and (12), we have

**Corollary 5:** Let $H$ be Hermitian and $P$ be positive semidefinite. Then

$$\sum_{i=1}^{k} \lambda_{n-i+1}(H) \lambda_i(P) \leq \sum_{i=1}^{k} \lambda_i(HP)$$

$$\leq \max_{U \succeq 0} \sum_{i=1}^{k} \lambda_i(H) \lambda_i(UPU^*) \tag{13}$$

and

$$\sum_{i=1}^{k} \lambda_i(H) \lambda_i(P) \geq \min_{U \succeq 0} \sum_{i=1}^{k} \lambda_{n-i+1}(H) \lambda_i(UPU^*) \tag{14}.$$

**Corollary 6:** Let $H$ and $P$ be both Hermitian. Then

$$\sum_{i=1}^{n} \lambda_{n-i+1}(H) \lambda_i(P) \leq \text{tr}(HP) \leq \sum_{i=1}^{n} \lambda_i(H) \lambda_i(P). \tag{15}$$

**Proof:** First, note that $U$ is unitary when $k = n$, and thus $\lambda_i(UPU^*) = \lambda_i(P)$. If $H$ is Hermitian and $P$ is positive semidefinite, then (15) follows from (13) or (14). If $P$ is Hermitian, we choose $\alpha$ so large that $P + \alpha I$ is positive semidefinite. A simple computation for $H$ and $P + \alpha I$ will yield (15).

The well known Hardy–Littlewood–Pólya rearrangement theorem (see, for instance, [10, p. 141]) follows from (15) at once by taking $H$ and $P$ to be $n \times n$ real diagonal matrices.

**Corollary 7:** Let $x_1 \geq x_2 \geq \cdots \geq x_n$ be a set of real numbers and $\pi$ be a permutation of $\{1, 2, \ldots, n\}$ such that $|x_{\pi(1)}| \geq |x_{\pi(2)}| \geq \cdots \geq |x_{\pi(n)}|$. Then $\pi$ is a permutation of $\{1, 2, \ldots, n\}$. Then, for $k = 1, 2, \ldots, n$

$$\sum_{i=1}^{k} |x_{n-i+1}| x_{\pi(i)} \leq \max_{p} \sum_{i=1}^{k} |x_{\pi(p)}| x_{\pi(i)}. \tag{16}$$

**Proof:** Let $H = \text{diag}(x_1, x_2, \ldots, x_n)$ and $P = \text{diag}(x_1, x_2, \ldots, x_n)$. Then, (16) follows from the first inequality of the theorem immediately.

We remark that Corollary 4 does not seem to follow from the Hardy–Littlewood–Pólya rearrangement theorem in which the upper limit for summation needs to be $n$. (Note: (2a) of A.3.a in [10, p. 141] is false. Take $(-3, -2)$ and $\{2, 1\}$.)

Since $\text{tr}(AP) = \text{tr}(PA)$ for any matrix $A$ and Hermitian matrix $B$, (15) yields (5).

For any complex matrix $X$, denote $|X| = (X^*X)^{1/2}$. Observing that $|X|$ is positive semidefinite and its eigenvalues are the singular values of $X$, we have the following.

**Corollary 8:** Let $A$ be Hermitian and $B$ be arbitrary. Then

$$\sum_{i=1}^{n} \lambda_{n-i+1}(A) \sigma_i(B) \leq \text{tr}(A|B|) \leq \sum_{i=1}^{n} \lambda_i(A) \sigma_i(B). \tag{17}$$

In particular, for arbitrary matrix $C$

$$\sum_{i=1}^{n} \lambda_{n-i+1}(\tilde{C}) \sigma_i(C) \leq \text{tr}(\tilde{C}|C|) \leq \sum_{i=1}^{n} \lambda_i(\tilde{C}) \sigma_i(C). \tag{18}$$

**Corollary 9:** Let $A$ and $B$ be complex matrices and $B = UDV^*$ be a singular value decomposition of $B$, where $U$ and $V$ are unitary, and $D$ is diagonal. Denote $S = UV^*$. Then

$$\sum_{i=1}^{n} \lambda_{n-i+1}(AS) \sigma_i(B) \leq \text{tr}(AB) \leq \sum_{i=1}^{n} \lambda_i(AS) \sigma_i(B). \tag{19}$$

Consequently, for real $A$ and $B$ (see [8])

$$\lambda_{\min}(AS) \sum_{i=1}^{n} \sigma_i(B) \leq \text{tr}(AB) \leq \lambda_{\max}(AS) \sum_{i=1}^{n} \sigma_i(B). \tag{20}$$

**Proof:** We make use the idea in the proof of [8]. Notice that

$$\text{tr}(AB) = \text{tr}(AUVDV^*) \leq \text{tr}(AUVD^*V^*) \leq \text{tr}(ASVDV^*).$$

Letting $H = \tilde{A}$ and $P = VDV^*$ in (15), we obtain (19) and thus (20).

**Corollary 10:** Let $I$ denote the number of eigenvalues of the $n \times n$ Hermitian matrix $H$ (including multiplicities) that lie in the open left half-plane. (If $I = n$ then $H$ is a stable matrix.) Let $P$ be an $n \times n$ positive semidefinite matrix. Then for every integer $k$, if $1 \leq k \leq l$, then

$$\sum_{1 \leq i < \cdots < k \leq n} \max_{1 \leq i \leq k} \lambda_{n-i+1}(H) \lambda_i(P)$$

$$= \sum_{i=1}^{k} \lambda_{n-i+1}(H) \lambda_{n-k+i}(P)$$

$$\leq \sum_{i=1}^{k} \lambda_i(HP).$$

**Proof:** Notice that if $1 \leq k \leq n$, all $\lambda_{n-k+1}(H) < 0$ for $i = 1, 2, \ldots, k$. Thus, the maximum value is attained by taking the $k$ smallest eigenvalues of $P$, that is, $\lambda_{n-k+1}(P) \geq \cdots \geq \lambda_{n-k+1}(P) \geq \lambda_0(P)$, i.e., $\lambda_{n-k+1}(P)$ for $i = 1, 2, \ldots, k$.

**Corollary 11:** Let $H$ be Hermitian and $P$ be positive semidefinite. Then

$$\max_{1 \leq i \leq n} \lambda_i(H) \lambda_i(P) \leq \lambda_1(H) \lambda_1(P) \leq \max_{u \neq 0} \lambda_1(U) \lambda_1(uPu^*)$$

and

$$\min_{1 \leq i \leq n} \lambda_i(H) \lambda_i(uPu^*) \leq \lambda_n(H) \lambda_1(P) \leq \min_{1 \leq i \leq n} \lambda_i(H) \lambda_i(P).$$

**Proof:** Putting $k = 1$ in Theorem 3 and Corollary 4 yields the previous inequalities, respectively.
As an immediate application to the Lyapunov equation (LE), if $A$ is stable and Hermitian and $X$ is positive semidefinite, then we have from Corollary 11

$$
\lambda_n(A)\lambda_n(X) \leq \lambda_1(AX) \leq \lambda_1(A)\lambda_n(X)
$$

and

$$
\lambda_n(A)\lambda_1(X) \leq \lambda_n(AX) \leq \lambda_1(A)\lambda_1(X).
$$

We remark that some singular value inequalities may be obtained for an arbitrary (rectangle) matrix $A$ by applying our theorem and corollaries to the partitioned Hermitian matrix $M = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$ and the partitioned positive semidefinite matrix $N = \begin{pmatrix} \sigma_1(A)I_n & A \\ A & \sigma_1(A)I_n \end{pmatrix}$. Note that the eigenvalues of $M$ are $\sigma_1(A), \ldots, \sigma_n(A)$, $-\sigma_n(A), \ldots, -\sigma_1(A)$, where $\sigma_1(A) \geq \cdots \geq \sigma_n(A)$ are the singular values of the matrix $A$.

IV. CONCLUSION

We have seen that our main result (Theorem 3) presents a set of general inequalities concerning the product of a positive semidefinite matrix and a Hermitian matrix. These inequalities give derivation of the inequalities in Section I, including the upper and lower bounds on the trace, by which the solutions to the matrix Riccati and Lyapunov equations have been estimated.

ACKNOWLEDGMENT

The first author would like to thank R. Horn for communication of [14] after completion of this note. Some related results are discussed and obtained in [14]. Both authors would like to thank the referees for carefully reading the manuscript.

REFERENCES


Exact Stability Analysis of 2-D Systems Using LMIs

Yoshio Ebihara, Yoshimichi Ito, and Tomomichi Hagiwara

Abstract—In this note, we propose necessary and sufficient conditions for the asymptotic stability analysis of two-dimensional (2-D) systems in terms linear matrix inequalities (LMIs). By introducing a guardian map for the set of Schur stable complex matrices, we first reduce the stability analysis problems into nonsingularity analysis problems of parameter-dependent complex matrices. Then, by means of the discrete-time positive real lemma and the generalized $S$-procedure, we derive LMI-based conditions that enable us to analyze the asymptotic stability in an exact (i.e., nonconservative) fashion. It turns out that, by employing the generalized $S$-procedure, we can derive smaller size of LMIs so that the computational burden can be reduced.

Index Terms—Linear matrix inequalities (LMIs), stability analysis, two-dimensional (2-D) systems.

I. INTRODUCTION

In this note, we address asymptotic stability analysis problems of two-dimensional (2-D) systems described by the Fornasini–Marchesini second model [6]

$$
x(i+1,j+1) = A_1x(i,j+1) + A_2x(i+1,j)
$$

$$
A_1,A_2 \in \mathbb{C}^{n \times n}. \quad (1)
$$

Precise definition of the asymptotic stability of the 2-D system was first made in [5]. Since then, various types of necessary and sufficient conditions have been proposed for the analysis of the asymptotic stability. We summarize some of them in the following proposition.

Proposition 1: [6] The following conditions are equivalent.

i) The 2-D system (1) is asymptotically stable.

ii) $\det(I_n - z_1A_1 - z_2A_2) \neq 0$ for all $(z_1,z_2) \in \mathbb{D} \times \mathbb{D}$ where $\mathbb{D}$ denotes the closure of the open unit disc $D$ on the complex plane.

iii) $\rho(A(\theta)) < 1$ for all $\theta \in [0, 2\pi]$ where $A(\theta) = A_1 + e^{i\theta}A_2$ and $\rho(\cdot)$ denotes the spectral radius.

Unfortunately, the conditions in ii) and iii) are not numerically tractable since they should be checked at infinitely many points over

Manuscript received April 3, 2004; revised March 22, 2005 and March 23, 2006. Recommended by Associate Editor D. Nesci. This work was supported in part by the Ministry of Education, Culture, Sports, Science, and Technology of Japan under Grant-in-Aid for Young Scientists (B) 15760314. Y. Ebihara and T. Hagiwara are with the Department of Electrical Engineering, Kyoto University, Kyoto 615-8510, Japan (e-mail: ebihara@kuee.kyoto-u.ac.jp). Y. Ito is with the Graduate School of Engineering, Osaka University, Osaka 565-0871, Japan.

Digital Object Identifier 10.1109/TAC.2006.880789

0018-9286/$20.00 © 2006 IEEE