The Schur complements of generalized doubly diagonally dominant matrices

Jianzhou Liu a, Yunqing Huang a, Fuzhen Zhang b, c, *

a Department of Mathematics, Xiangtan University, Xiangtan, Hunan 411105, China
b Department of Math, Science, and Technology, Nova Southeastern University, 3301 College Avenue, Fort Lauderdale, FL 33314, USA
c Department of Mathematics, Shenyang Normal University, Shenyang, China

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Abstract

As is known, the Schur complements of diagonally dominant matrices are diagonally dominant; the same is true of doubly diagonally dominant matrices. The purpose of this paper is to extend the results to the generalized doubly diagonally dominant matrices (a proper subset of H-matrices); that is, we show that the Schur complement of a generalized doubly diagonally dominant matrix is a generalized doubly diagonally dominant matrix.

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* Corresponding author. Address: Department of Math, Science, and Technology, Nova Southeastern University, 3301 College Avenue, Fort Lauderdale, FL 33314, USA. Tel.: +1-954-262-8317; fax: +1-954-262-3931.
E-mail addresses: liujz@xtu.edu.cn (J. Liu), zhang@nova.edu (F. Zhang).

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1. Introduction and notation

Given a matrix family, it is always interesting to know whether some important properties or structures of the family of the matrices are inherited by their submatrices or by the matrices associated with the original matrices. It is known that the principal submatrices and the Schur complements of positive semidefinite matrices are positive semidefinite; the same is true of $M$-matrices, $H$-matrices, and of inverse $M$-matrices (see, e.g., [11]).

Carlson and Markham [3] showed that the Schur complements of strictly diagonally dominant matrices are diagonally dominant. The very property has been repeatedly used for the convergence of the Gauss–Seidel iterations in numerical analysis (see, e.g., [12, p. 58] or [7, p. 508]).

As the Geršgorin discs play a key role in locating the spectra of square matrices in the complex plane and as it ensures the nonsingularity of the strictly diagonally dominant matrices, the Cassini ovals give rise to the doubly diagonally dominant matrices. A diagonally dominant matrix is automatically doubly diagonally dominant, but not conversely. Both matrices have been used in the study of $M$- and $H$-matrices (see, e.g., [5]). Ikramov [10], also Li and Tsatsomeros [13] independently, proved that the Schur complements of doubly diagonally dominant matrices are doubly diagonally dominant.

We shall extend these results to generalized doubly diagonally dominant matrices, which were studied by Gao and Wang [6].

To begin with, let $\mathbb{C}^{n \times n}$ denote the set of all $n \times n$ complex matrices and $\pi = \{1, 2, \ldots, n\}$. Let $\alpha$ be a proper subset of $\pi$ and denote by $|\alpha|$ the cardinality of $\alpha$ and by $\alpha' = \pi - \alpha$ the complement to $\alpha$ in $\pi$. The elements of $\alpha$ and $\alpha'$ are both conventionally arranged in increasing order. For nonempty index sets $\alpha, \beta \subseteq \pi$, we write $A(\alpha, \beta)$ to mean the submatrix of $A \in \mathbb{C}^{n \times n}$ lying in the rows indexed by $\alpha$ and the columns indexed by $\beta$. $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$.

Assume that $A(\alpha)$ is nonsingular. The Schur complement of $A$ with respect to $A(\alpha)$, denoted by $A/A(\alpha)$ or simply $A/\alpha$, is defined to be

$$A(\alpha') - A(\alpha', \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha').$$  \hfill (1)

The Schur complements (see, e.g., [8, p. 22] or [17, p. 175]) and their extensions—generalized Schur complements (see, e.g., [16]), which are defined for positive semidefinite matrices $A$ when $A(\alpha)$ is singular by replacing $[A(\alpha)]^{-1}$ in (1) with the Moore–Penrose inverse $[A(\alpha)]^+$, have various applications in many aspects of matrix theory (see, e.g., [18]), in applied math (see, e.g., [1,2]), and in statistics (see, e.g., [14] or [15]). We shall confine ourselves to the nonsingular $A(\alpha)$ as far as $A/A(\alpha)$ is concerned.

A remarkable Schur determinantal formula says (see, e.g., [17, p. 175])

$$\det(A/A(\alpha)) = \frac{(\det A)}{(\det A(\alpha))}.$$
Let $A = (a_{ij})$ be an $n \times n$ matrix, $n \geq 2$. Denote for each $i = 1, 2, \ldots, n$,

$$P_i(A) = \sum_{j=1, j\neq i}^n |a_{ij}|.$$  

Recall that $A$ is (row) diagonally dominant if for all $i = 1, 2, \ldots, n$,

$$|a_{ii}| \geq P_i(A).$$  

(2)

$A$ is further said to be strictly diagonally dominant if all the strict inequalities in (2) hold. It is well known that strictly diagonally dominant matrices are nonsingular (by the Gersgorin Theorem). Obviously the principal submatrices of strictly diagonally dominant matrices are strictly diagonally dominant and thus nonsingular.

A doubly diagonally dominant matrix is a matrix such that for all $i \neq j$,

$$|a_{ii}| |a_{jj}| \geq P_i(A) P_j(A)$$  

(3)

and that $A$ is strictly doubly diagonally dominant if all the strict inequalities in (3) hold. Strictly doubly diagonally dominant matrices are nonsingular (by the Brauer theorem; see, e.g., [8, p. 381]), and so are the principal submatrices. In addition, if $A$ is doubly diagonally dominant, there exists at most one index $i_0$ such that

$$|a_{ii_0}| < P_{i_0}(A).$$

We shall extend the definition of a doubly diagonally dominant matrix to a generalized doubly diagonally dominant matrix and show, as for the doubly diagonally dominant case, that the generalized doubly diagonally dominant matrix preserves the property: its Schur complement is the same kind.

For $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, the rows of $A$ may be divided into two groups: the ones that are (row) diagonally dominant (i.e., satisfying (2)) and the ones that are not. Let $q = \{i \in n \mid |a_{ii}| > P_i(A)\}$. If $q = n$, then $A$ is strictly diagonally dominant. We are interested in the matrices for which at least one row is diagonally dominant; i.e., $|a_{ii}| > P_i(A)$ for at least one $i \in n$; namely, $q \neq \emptyset$.

We call $A$ a generalized doubly diagonally dominant matrix if there exist proper subsets $n_1, n_2$ of $n$ such that $n_1 \cap n_2 = \emptyset, n_1 \cup n_2 = n$ and

$$(|a_{ii}|-\alpha_s)(|a_{jj}|-\beta_j) \geq \beta_i \alpha_j$$  

(4)

for all $i \in n_1$ and $j \in n_2$, where, with $s = i$ or $j$,

$$\alpha_s = \sum_{i \in n_1 \atop i \neq s} |a_{is}|, \quad \beta_s = \sum_{i \in n_2 \atop i \neq s} |a_{is}|.$$  

Here the $\alpha_s$ and $\beta_s$ may be interpreted as the sums of the absolute values of the nondiagonal elements in row $s$ that fall in the columns $n_1$ and $n_2$, respectively. If $n_1$ or $n_2$ contains a single element, say $n_1 = \{s_0\}$, then we assume $\alpha_{s_0} = 0$ in convention. Similarly $\beta_{s_0} = 0$ if $n_2 = \{s_0\}$. When $n = 2$, a generalized doubly diagonally dominant matrix is then simply a doubly diagonally dominant matrix.
dominant matrix is nothing but a doubly diagonally dominant matrix. Moreover, a matrix $A$ satisfying (4) may not be generalized doubly diagonally dominant for another pair subsets $n_1$ and $n_2$.

It is readily seen that for $i \in n_1$ and for $j \in n_2$

$$\alpha_i = P_i(A(n_1)), \quad \beta_j = P_j(A(n_2)).$$

In addition,

$$P_s(A) = \alpha_s + \beta_s.$$ 

If we denote

$$\delta_s = |a_{ss}| - P_s(A),$$

then the inequalities (4) may be rewritten as, for $i \in n_1$ and for $j \in n_2$,

$$(\delta_i + \beta_i)(\delta_j + \alpha_j) \geq \beta_i\alpha_j.$$ (5)

We call $A$ a strictly generalized doubly diagonally dominant matrix if all the strict inequalities in (4) hold.

Assuming that the matrix order is $n \geq 2$, we adopt the following notations:

$D_n$ for diagonally dominant matrices;

$SD_n$ for strictly diagonally dominant matrices;

$DD_n$ for doubly diagonally dominant matrices;

$SDD_n$ for strictly doubly diagonally dominant matrices;

$GDD_n^{n_1,n_2}$ for generalized doubly diagonally dominant;

$SGDD_n^{n_1,n_2}$ for strictly generalized doubly diagonally dominant.

Inspecting (4), if $n_2$ (or $n_1$, similarly) is empty, all $\beta_i$ and $\beta_j$ vanish, we may interpret (4) as $|a_{ii}| - \alpha_i \geq 0$, or $A$ is diagonally dominant. Conventionally, we write $GDD_n^{0,n} = GDD_n^{n,0} = D_n$ and $SGDD_n^{0,n} = SGDD_n^{n,0} = SD_n$.

We will sometimes suppress the subscript $n$ and the superscripts $n_1$ and $n_2$ unless a confusion is caused. So when we write $A \in GDD$ or we say $A$ is GDD, for instance, we mean that $A$ is an $n \times n$ generalized doubly diagonally dominant matrix with respect to some subsets $n_1$ and $n_2$ of $n$.

We observe that if $A$ is GDD then so is a principal submatrix of $A$. To be precise, let $A \in GDD_n^{n_1,n_2}$ and $m \subseteq n$. Then $A(m) \in GDD_m^{m_1,m_2}$ or $SD_m^{m_1,m_2}$. The same is true for SGDD matrices. Besides, if $A$ is SGDD, then $A$ is an $H$-matrix [6, Theorem 1], so $A$ is nonsingular. Notice that an $H$-matrix is not necessarily SGDD as the following example shows. Take

$$A = \begin{pmatrix} 3 & 2 & 0 & 4 \\ 0 & 6 & 3 & 4 \\ 1 & 2 & 9 & 0 \\ 0 & 2 & 3 & 12 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{pmatrix}.$$

$A$ is an $H$-matrix, but $A$ is not SGDD for any subsets $n_1$ and $n_2$ of $\{1, 2, 3, 4\}$. 

2. Propositions and lemmas

The following propositions are consequences derived from the definition and inequalities (4). It is obvious from inequalities (5) that strictly diagonally dominant matrices are strictly generalized doubly diagonally dominant. We further observe that the generalized doubly diagonally dominant matrix is indeed a generalization of the doubly diagonally dominant matrix.

Proposition 1. If \( A \in \text{SDD} \) (or \( \text{DD} \)), then \( A \in \text{SGDD} \) (resp. \( \text{GDD} \)).

Proof. If \( A \in \text{SDD} \) and \( A \notin \text{SD} \), then there exists one index \( i_0 \) such that
\[
|a_{ii_0}| \leq P_{i_0}(A) \quad \text{and} \quad |a_{ii}| > P_{i}(A) \quad \text{for all} \ i \neq i_0.
\]
Take \( n_1 = n - \{i_0\} \), \( n_2 = \{i_0\} \). Then, for \( i \in n_1 \) and \( j \in n_2 \),
\[
\alpha_i = \sum_{t \in n_1 \backslash \{i\}} |a_{it}|, \quad \alpha_j = P_{i_0}(A), \quad \beta_i = |a_{ii_0}|, \quad \beta_j = 0.
\]
We compute
\[
(|a_{ii_0}| - \alpha_i)(|a_{jj}| - \beta_j) - \beta_i \alpha_j
\]
to get
\[
\left(|a_{ij}| - \sum_{\substack{t \in n_1 \backslash \{i\} \atop t \neq i_0}} |a_{it}| \right)|a_{ii_0}| - |a_{ii_0}|P_{i_0}(A)
\]
\[
= |a_{ij}||a_{ii_0}| - |a_{ii_0}| \sum_{t \neq i_0} |a_{it}| - |a_{ii_0}|P_{i_0}(A)
\]
\[
\geq |a_{ij}||a_{ii_0}| - P_{i_0}(A) \sum_{t \neq i_0} |a_{it}| - |a_{ii_0}|P_{i_0}(A)
\]
\[
= |a_{ij}||a_{ii_0}| - P_i(A)P_{i_0}(A) > 0.
\]
So \( A \) is SGDD by definition. The DD or GDD case is similarly proven. □

We point out that Proposition 1 is essentially the same as Remark 2.1 in [6]. The following diagram shows the set inclusions:

\[
\begin{array}{ccc}
\text{SD} & \subset & \text{SDD} \\
\cap & \cap & \cap \\
\text{D} & \subset & \text{DD} \subset \text{GDD}.
\end{array}
\]

In the next proposition we show that if \( A \) is strictly generalized doubly diagonally dominant with respect to the subsets \( n_1 \) and \( n_2 \) of \( n \), then the submatrices \( A(n_1) \) and \( A(n_2) \) of \( A \) are diagonally dominant. We show the case where \( A \) is “strict”. The regular “generalized” case is about the same.
Proposition 2. Let $A \in \text{SGDD}_{n_1,n_2}^n$. Then $A(n_1)$ and $A(n_2)$ are in SD.

Proof. If there exists some $i_0 \in n_1$ such that
\[ |a_{i_0i_0}| - \alpha_{i_0} < 0, \]
then by (4), there must exist some $j_0 \in n_2$ such that
\[ |a_{j_0j_0}| - \beta_{j_0} < 0. \]
On the other hand, by definition, $A \in \text{SGDD}_{n_1,n_2}^n$ implies that for any $i \in n_1$,
\[ (|a_{ii}| - \alpha_i)(|a_{j_0j_0}| - \beta_{j_0}) > \beta_i \alpha_{j_0} \geq 0, \]
which yields for any $i \in n_1$, $|a_{ii}| < \alpha_i \leq P_i(A)$. Similarly, for any $j \in n_2$, $|a_{jj}| < \beta_j \leq P_j(A)$. Thus $g = \emptyset$ and this contradicts our assumption that $g \neq \emptyset$. We conclude that for all $i \in n_1$ and $j \in n_2$
\[ |a_{ii}| > \alpha_i, \quad |a_{jj}| > \beta_j \]
equivalently
\[ |a_{ii}| > P_i(A(n_1)), \quad |a_{jj}| > P_j(A(n_2)). \quad \square \]

Our next result, which may be considered as a modified version of the previous proposition, shows that when $A$ is a strictly generalized diagonally dominant matrix with respect to proper subsets $n_1$ and $n_2$ of $n$, then either
\[ |a_{ii}| > P_i(A) \quad \text{for all } i \in n_1, \quad \text{or} \quad |a_{jj}| > P_j(A) \quad \text{for all } j \in n_2. \]

Proposition 3. Let $A \in \text{SGDD}_{n_1,n_2}^n$. Then $n_1 \subseteq g$ or $n_2 \subseteq g$.

Proof. Let $n_1$, $n_2$ be proper subsets of $n$ such that $n_1 \cap n_2 = \emptyset$ and $n_1 \cup n_2 = n$. If $|a_{ii}| > P_i(A)$ for all $i \in n_1$, then $n_1 \subseteq g$; if $|a_{jj}| > P_j(A)$ for all $j \in n_2$, then $n_2 \subseteq g$. Suppose, otherwise, that there exist $i_0 \in n_1$ and $j_0 \in n_2$ such that
\[ |a_{i_0i_0}| \leq P_{i_0}(A) \quad \text{and} \quad |a_{j_0j_0}| \leq P_{j_0}(A). \]
Then
\[ |a_{i_0i_0}| - \alpha_{i_0} \leq \beta_{i_0} \quad \text{and} \quad |a_{j_0j_0}| - \beta_{j_0} \leq \alpha_{j_0}. \]
Thus
\[ (|a_{i_0i_0}| - \alpha_{i_0})(|a_{j_0j_0}| - \beta_{j_0}) \leq \beta_{i_0} \alpha_{j_0}, \]
and $A$ is not SGDD. Therefore, if $A \in \text{SGDD}_{n_1,n_2}^n$, then $n_1 \subseteq g$ or $n_2 \subseteq g$. \quad \square

We continue with some results as lemmas on comparison matrices as preparation for our next section. The comparison matrix has been heavily used in the study of $H$-matrices and $M$-matrices (see, e.g., [9, p. 123]).
The comparison matrix \( \mu(A) = (c_{ij}) \) of a given \( A = (a_{ij}) \) is defined to be

\[
c_{ij} = \begin{cases} 
|a_{ij}| & \text{if } i = j, \\
-|a_{ij}| & \text{if } i \neq j.
\end{cases}
\]

We say that matrix \( A \) is an \( H \)-matrix if \( \mu(A) \) is an \( M \)-matrix. Recall that a square matrix \( M \) is an \( M \)-matrix if it can be written in the form \( M = \alpha I - P \), where \( P \) is a nonnegative matrix and \( \alpha > \rho(M) \), the spectral radius of \( M \). Denote by \( \mathbb{H}_n \) and \( \mathbb{M}_n \) the sets of \( n \times n \) \( H \)- and \( M \)-matrices, respectively.

For any matrix \( A = (a_{ij}) \), we denote \( |A| = (|a_{ij}|) \). If the entries of the matrix \( A \) are all nonnegative, then we write \( A \geq 0 \). For real matrices \( A \) and \( B \) of the same size, if \( A - B \) is a nonnegative matrix, we write \( A \geq B \).

**Lemma 1.** Let \( A \in \mathbb{C}^{n \times n} \) and \( B \in \mathbb{M}_n \). If \( \mu(A) \geq B \), then \( A \in \mathbb{H}_n \) and

\[
B^{-1} \geq |A^{-1}| \geq 0.
\]

**Proof.** See [5, Theorems 4.2 and 4.6] or [9, p. 117 or 131]. \( \square \)

It follows immediately from Lemma 1 that

\[
A \in \mathbb{H}_n \Rightarrow [\mu(A)]^{-1} \geq |A^{-1}|.
\]

**Lemma 2.** Let \( A \in \text{SD}_n \), \( \text{SDD}_n \), or \( \text{SGDD}_n \). Then \( \mu(A) \in \mathbb{M}_n \); i.e., \( A \in \mathbb{H}_n \).

**Proof.** See [5, 4.3, 6.9; 6; 9, p. 114; 13, Theorem 2.1]. \( \square \)

With the example in Section 1, we have the proper inclusion \( \text{SGDD}_n \subseteq \mathbb{H}_n \). As is known, the Schur complement of an \( H \)-matrix is an \( H \)-matrix. Thus the Schur complement of a \( \text{SGDD} \) matrix is an \( H \)-matrix, which does not have to be \( \text{SGDD} \) at this point. This is what we are to prove in the next section.

**Lemma 3.** Let \( A \in \text{SD}_n \) and \( m \) be a proper subset of \( n \). Then \( A/m \in \text{SD}_{n-[m]} \).

**Proof.** This is the main result in [3]. \( \square \)

**Lemma 4 (Quotient formula).** Let \( A \) be a square matrix. If \( B \) is a nonsingular principal submatrix of \( A \) and \( C \) is a nonsingular principal submatrix of \( B \). Then \( B/C \) is a nonsingular principal submatrix of \( A/C \) and

\[
A/B = (A/C)/(B/C).
\]

**Proof.** See [4] or [19]. \( \square \)
3. Main results

Our first theorem states that if $A$ is a strictly generalized doubly diagonally dominant matrix with respect to subsets $n_1$ and $n_2$ of $n$ and if $m \subset n$ contains $n_1$ or $n_2$, then the Schur complement $A/m$ is strictly diagonally dominant.

**Theorem 1.** Let $A \in \text{SGDD}^n_{n_1,n_2}$. If $n_1 \subseteq m \subseteq n$ or $n_2 \subseteq m \subseteq n$, then $A/m \in \text{SD}_{n-|m|}$.

**Proof.** Let $n_1 = \{i_1, \ldots, i_k\}$ and $n_2 = \{j_1, \ldots, j_l\}$, where $k + l = n$. We assume $n_1 \subseteq g$ (or $n_2 \subseteq g$) by Proposition 3 and break down the proof into three cases.

(i) $n_1 = m$: Since $A$ is SGDD, we have by (4), for any $i_1 \in n_1$, $j \in n_2$,

$$|(a_{i_1j_1} - \alpha_{i_1})(a_{jj} - \beta_j)| > \alpha_j \beta_{i_1}$$

or

$$|a_{jj} - \beta_j| > \frac{\beta_{i_1}}{|a_{i_1j_1}| - \alpha_{i_1}} \alpha_j.$$  \hspace{1cm} (7)

So for $j \in n_2$,

$$|a_{jj} - \beta_j| > \max_{i \in n_1} \frac{\beta_{i}}{|a_{ii}| - \alpha_{i}} \alpha_j.$$  \hspace{1cm} (8)

Note that $A(m) = A(n_1)$ is nonsingular. Write

$$x = [\mu[A(m)]]^{-1} \begin{pmatrix} \sum_{s=1}^{l} |a_{i_1j_s}| \\ \vdots \\ \sum_{s=1}^{l} |a_{i_kj_s}| \end{pmatrix} \quad \text{or} \quad \mu[A(m)]x = \begin{pmatrix} \sum_{s=1}^{l} |a_{i_1j_s}| \\ \vdots \\ \sum_{s=1}^{l} |a_{i_kj_s}| \end{pmatrix}.$$

Letting $x_v = \max\{x_1, \ldots, x_k\}$, where $x_i$ is the $i$th component of $x$, we obtain

$$\sum_{s=1}^{l} |a_{i_sj_s}| = |a_{i_1i_v}|x_v - \sum_{r \neq v}^{k} x_r |a_{i_ri_v}| \geq x_v \left( |a_{i_1i_v}| - \sum_{r \neq v}^{k} |a_{i_ri_v}| \right),$$

which gives

$$\frac{\sum_{s=1}^{l} |a_{i_sj_s}|}{|a_{i_1i_v}| - \sum_{r \neq v}^{k} |a_{i_ri_v}|} = \frac{\beta_{i_v}}{|a_{i_1i_v}| - \alpha_{i_v}} \geq x_v.$$  \hspace{1cm} (8)

Thus

$$\max_{i \in n_1} \frac{\beta_i}{|a_{ii}| - \alpha_i} \geq x_v.$$  \hspace{1cm} (8)
Denote the \((s, t)\)-entry of \(A/m\) by \((a'_{j_t, j_s})\). Then for \(t = 1, 2, \ldots, l\), we have

\[
|a'_{j_t, j_s}| - \sum_{s=1}^{l} |a'_{j_t, j_s}|
\]

\[
= a_{j_t, j_s} - (a_{j_t, j_1}, \ldots, a_{j_t, j_l})[A(m)]^{-1} \begin{pmatrix} a_{i_1, j_s} \\ \vdots \\ a_{i_l, j_s} \end{pmatrix}
\]

\[
= \sum_{s=1}^{l} \left| a_{j_t, j_s} - (a_{j_t, j_1}, \ldots, a_{j_t, j_l})[A(m)]^{-1} \begin{pmatrix} a_{i_1, j_s} \\ \vdots \\ a_{i_l, j_s} \end{pmatrix} \right|
\]

\[
\geq \left| a_{j_t, j_s} - (a_{j_t, j_1}, \ldots, a_{j_t, j_l})[A(m)]^{-1} \begin{pmatrix} a_{i_1, j_s} \\ \vdots \\ a_{i_l, j_s} \end{pmatrix} \right| \quad \text{(by (6))}
\]

\[
\geq \left| a_{j_t, j_s} - (a_{j_t, j_1}, \ldots, a_{j_t, j_l})[\mu A(m)]^{-1} \begin{pmatrix} a_{i_1, j_s} \\ \vdots \\ a_{i_l, j_s} \end{pmatrix} \right| \quad \text{(by (8))}
\]

\[
\geq \left| a_{j_t, j_s} - \sum_{s=1}^{l} \frac{\beta_i}{\max_{i \in n_1} |a_{i,j_s}| - \alpha_i} \right|
\]

\[
= |a_{j_t, j_s}| - \sum_{s=1}^{l} |a_{j_t, j_s}| - \max_{i \in n_1} \frac{\beta_i}{|a_{i,j_s}| - \alpha_i} \quad \text{(by (7)).}
\]
It follows that 
\[ A/m \in \text{SD}_{n-|m|}. \]

(ii) \( n_1 \subset m \subset n \): By Lemma 3, \( A/n_1 \in \text{SD}_{n-|n_1|} \). By Lemma 4 and using the fact that if a matrix \( X \) is SD then so is its Schur complement, we have 
\[ A/m = (A/n_1)/(A(m)/n_1) \in \text{SD}. \]

(iii) The case where \( n_2 \subseteq m \subset n \) is similarly proven. \( \square \)

The following example shows that the condition that \( n_1 \subseteq m \) or \( n_2 \subseteq m \) is necessary for the statement to hold. Take \( n_1 = \{1, 2, 3\} \), \( n_2 = \{4\} \), and \( m = \{1, 2\} \) for 
\[ A = \begin{pmatrix}
2 & 1 & 0 & 0 \\
1 & 2 & 0 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & 3 & 1
\end{pmatrix} \in \text{SGDD}_{n_1, n_2}^{n_1, n_2}. \]

But 
\[ A/m = \begin{pmatrix}
2 & 1 \\
3 & 1
\end{pmatrix} \notin \text{SD}_2. \]

Our next result asserts that if \( A \) is SGDD then so is the Schur complement \( A/m \); that is, the class SGDD is closed under the Schur complement.

**Theorem 2.** Let \( A \in \text{SGDD}_{n_1, n_2}^n \). Then for any proper subset \( m \) of \( n \), 
\[ A/m \in \text{SGDD}_{n-|m|}^{n_1, n_2-m}. \]

**Proof.** Let \( n_1 = \{i_1, \ldots, i_k\}, n_2 = \{j_1, \ldots, j_l\}, l + k = n \). We show the theorem in three steps: \( m \) is a singleton; \( m \subseteq n_1 \) or \( m \subseteq n_2 \); \( m \not\subset n_1 \) and \( m \not\subset n_2 \).

(i) Consider the case where \( m \) contains only one element. Assume \( m \subseteq n_1 \).

If \( m = n_1 = \{i_1\} \), by Theorem 1, we have 
\[ A/m \in \text{SD}_{n-1} = \text{SGDD}_{n_1, n_2}^{\emptyset, n_2}. \]

If \( m = \{i_1\} \subset n_1 \), for any fixed \( j_a \in n_2 \) and \( i_y \in n_1 \setminus \{i_1\} \), let 
\[ a_1 = \begin{pmatrix}
|a_{i_1i_1}| & -\sum_{\substack{i \neq j_a \\text{ or} \ i \notin n_1 \\text{ or} \ i \notin n_2}} |a_{i_1i_1}| & -\sum_{j_a \in n_2} |a_{i_1j_a}| \\
-|a_{i_1i_1}| & |a_{i_1i_y}| - \sum_{\substack{i \neq j_a \\text{ or} \ i \notin n_1 \\text{ or} \ i \notin n_2}} |a_{i_1i_1}| & -\sum_{j_a \in n_2} |a_{i_1j_y}| \\
-|a_{j_a1}| & -\sum_{\substack{i \neq j_a \\text{ or} \ i \notin n_1 \\text{ or} \ i \notin n_2}} |a_{j_a1}| & |a_{j_az} - \sum_{j_a \in n_2} |a_{j_a1}| \end{pmatrix}. \]
Since $A \in \text{SGDD}_{n_1,n_2}$, we have, by (4),
\[
\begin{pmatrix}
|a_{i_1i_1}| - \sum_{r \neq i_1 \atop j \in n_1} |a_{i_1j}| & |a_{j_1j_1}| - \sum_{s \neq j_1 \atop j_1 \in n_2} |a_{j_1j_1}| \\
\end{pmatrix}
> \left( \sum_{j_1 \in n_2} |a_{i_1j_1}| \right)
\begin{pmatrix}
|a_{j_1i_1}| + \sum_{r \neq j_1 \atop i_1 \in n_1} |a_{j_1i_1}| \\
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
|a_{i_1i_1}| - \sum_{r \neq i_1 \atop j \in n_1} |a_{i_1j}| & |a_{j_1j_1}| - \sum_{s \neq j_1 \atop j_1 \in n_2} |a_{j_1j_1}| \\
\end{pmatrix}
> \left( \sum_{j_1 \in n_2} |a_{i_1j_1}| \right)
\begin{pmatrix}
|a_{j_1i_1}| + \sum_{r \neq j_1 \atop i_1 \in n_1} |a_{j_1i_1}| \\
\end{pmatrix}
\]

By Proposition 3, we have (for the case $n_1 \subseteq g$)
\[
|a_{i_1i_1}| > \sum_{r \neq i_1 \atop j \in n_1} |a_{i_1j}| + \sum_{j_1 \in n_2} |a_{i_1j_1}|
\]
and
\[
|a_{i_1i_1}| - \sum_{r \neq i_1 \atop j \in n_1} |a_{i_1j}| > |a_{i_1i_1}| + \sum_{j_1 \in n_2} |a_{i_1j_1}|
\]
or (for the case $n_2 \subseteq g$)
\[
|a_{j_1j_1}| - \sum_{s \neq j_1 \atop j_1 \in n_2} |a_{j_1j_1}| > |a_{j_1i_1}| + \sum_{r \neq j_1 \atop i_1 \in n_1} |a_{j_1i_1}|.
\]

Thus
\[
A_1 \in \text{SD}_3 \subseteq \text{SGDD}_3.
\]

By Lemma 2, we have $A_1 = \mu(A_1) \in \mathcal{M}_3$. Moreover,
\[
\det A_1 > 0. \quad (9)
\]

We now show that $A/m$ satisfies the inequalities (4). Identify the $(s, t)$-entry of $A/m$ by $a'_{j_s,j_t}$. Upon computation, we have
\[
\begin{align*}
&\left\{ \left| a'_{i,j_n} \right| - \sum_{r \neq j_n \atop i, j \in n_1} \left| a'_{i,j} \right| \right\} \cdot \left\{ \left| a'_{j_n,j_n} \right| - \sum_{r \neq j_n \atop j_n \in n_2} \left| a'_{j_n,j} \right| \right\} \\
&\quad - \left\{ \sum_{j_n \in n_2} \left| a'_{i,j_n} \right| \right\} \cdot \left\{ \sum_{r \neq i \atop i \in n_1} \left| a'_{i,j} \right| \right\} \\
&= \left( a_{i,j} - \frac{a_{i,i} a_{i,j}}{a_{i,i}} \right) - \sum_{r \neq i \atop i \in n_1} \left( a_{i,i} - \frac{a_{i,i} a_{i,j}}{a_{i,i}} \right) \\
&\quad \times \left( a_{j_n,j_n} - \frac{a_{j_n,j_n} a_{j_n,j_n}}{a_{j_n,j_n}} \right) - \sum_{r \neq j_n \atop j_n \in n_2} \left( a_{j_n,j_n} - \frac{a_{j_n,j_n} a_{j_n,j_n}}{a_{j_n,j_n}} \right) \\
&\quad \times \left( a_{j_n,j_n} - \frac{a_{j_n,j_n} a_{j_n,j_n}}{a_{j_n,j_n}} \right) \\
&\quad \times \left( a_{i,i} - \frac{a_{i,i} a_{i,i}}{a_{i,i}} \right) - \sum_{r \neq i \atop i \in n_1} \left| a_{i,i} a_{i,i} \right| \\
&\quad \times \left( a_{j_n,j_n} - \frac{a_{j_n,j_n} a_{j_n,j_n}}{a_{j_n,j_n}} \right) - \sum_{r \neq j_n \atop j_n \in n_2} \left| a_{j_n,j_n} a_{j_n,j_n} \right| \\
&\quad \times \left( a_{j_n,j_n} - \frac{a_{j_n,j_n} a_{j_n,j_n}}{a_{j_n,j_n}} \right) \\
&\quad \times \left( a_{i,i} - \frac{a_{i,i} a_{i,i}}{a_{i,i}} \right) - \sum_{r \neq i \atop i \in n_1} \left| a_{i,i} a_{i,i} \right| \\
&\quad \times \left( a_{j_n,j_n} - \frac{a_{j_n,j_n} a_{j_n,j_n}}{a_{j_n,j_n}} \right) - \sum_{r \neq j_n \atop j_n \in n_2} \left| a_{j_n,j_n} a_{j_n,j_n} \right| \\
&\quad \times \left( a_{j_n,j_n} - \frac{a_{j_n,j_n} a_{j_n,j_n}}{a_{j_n,j_n}} \right) \\
&= \det[A_1/|a_{i,i}|] = \frac{1}{|a_{i,i}|} \det A_1 > 0 \quad \text{(by \(9\))}.
\end{align*}
\]
So for any $i_1 \in \Omega_1$,

$$A/[i_1] \in \text{SGDD}^{n_1-\{i_1\},n_2}_{n-1}.$$  

(ii) If $m$ contains more than one element and if $m \subseteq \Omega_1$ or $m \subseteq \Omega_2$, we take $i_1 \in m$. Then by the quotient formula and by induction on the order of the matrices, we have $A/m = (A/[i_1])/(A(m)/[i_1]) \in \text{SGDD}$.

(iii) If $m$ is contained neither in $\Omega_1$ nor in $\Omega_2$, in a similar way,

$$A/m = (A/(m \cap \Omega_1))/(A(m)/(m \cap \Omega_1)) \in \text{SGDD}. \quad \square$$

Finally we remark that we may slightly relax the “strict” condition so that our theorems hold for some matrices with certain involved nonsingular principal submatrices. This is done by a usual trick—continuity argument. We omit further discussions on this.

References


