Matrix Inequalities by Means of Block Matrices

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We first show a weak log-majorization inequality of singular values for partitioned positive semidefinite matrices which will imply some existing results of a number of authors, then present some basic matrix inequalities and apply them to obtain a number of matrix inequalities involving sum, ordinary product and Hadamard product.

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1. INTRODUCTION

One of the most useful tools for deriving matrix inequalities is to utilize block matrices; usually they are $2 \times 2$ in most applications. In this paper, we shall show a weak log-majorization inequality of singular values for partitioned positive semidefinite matrices, from which some classical and recent results of Bhatia and Kittaneh [4], Wang, Xi and Zhang [12], and Zhan [13] will follow. We shall also develop a new technique that is complementary to the Schur complement; while by making use of Schur complements, a number of determinantal, trace, and other inequalities are exhibited in [16]. With the new technique we add more inequalities to these in [16].

We denote the eigenvalues of an $n \times n$ complex matrix $X$ by $\lambda_i(X)$, $i = 1, 2, \ldots, n$, and arrange them in modulus decreasing order $|\lambda_1(X)| \geq |\lambda_2(X)| \geq \cdots \geq |\lambda_n(X)|$.

The singular values of an $m \times n$ matrix $X$ are denoted by $\sigma_1(X), \ldots, \sigma_n(X)$ and are also arranged in decreasing order. Note that $\sigma_i(X) = \lambda_i(|X|)$ for each $i$, where $|X| = (X^*X)^{1/2}$. We further write

$$\lambda(X) = (\lambda_1(X), \lambda_2(X), \ldots, \lambda_n(X)), \quad \sigma(X) = (\sigma_1(X), \sigma_2(X), \ldots, \sigma_n(X)).$$

The leading principal $k \times k$ submatrix of a matrix $X$ is denoted by $[X]_k$. 

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Write $X \geq 0$ if $X$ is a positive semidefinite matrix and $X \geq Y$ if $X$ and $Y$ are Hermitian matrices such that $X - Y \geq 0$. The strict inequality $X > 0$ denotes the positive definiteness of $X$. Let $X \circ Y = (x_{ij}y_{ij})$ be the Hadamard (Schur) product of matrices $X$ and $Y$ of the same size and $X^*$ the conjugate transpose of $X$.

For complex vector $x = (x_1, x_2, \ldots, x_n)$, we denote $|x| = (|x_1|, |x_2|, \ldots, |x_n|)$, and for vectors $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ with nonnegative components in decreasing order, we write $\log x \prec_w \log y$ to mean
\[
\prod_{i=1}^{k} x_i \leq \prod_{i=1}^{k} y_i, \quad k = 1, 2, \ldots, n.
\]
As is well known, $\log x \prec_w \log y$ yields $x \prec_w y$ (see, e.g., [3, p. 42]). The latter means
\[
\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad k = 1, 2, \ldots, n.
\]
The subscript “$w$” is dropped off in either $\prec_w$ if equality holds when $k = n$.

### 2. A WEAK LOG-MAJORIZATION INEQUALITY

**Theorem 1** Let $A$, $B$, and $C$ be complex matrices such that
\[
\begin{pmatrix}
  A & B \\
  B^* & C
\end{pmatrix} \geq 0,
\]
where $A$ is $m \times m$, $C$ is $n \times n$, and $B$ is $m \times n$. Let $\text{rank}(B) = r$. Then
\[
\log \sigma(B) \prec_w \log \mu,
\]
where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ with $\mu_i = \max\{\lambda_i(A), \lambda_i(C)\}$ if $i \leq r$, $0$ otherwise. Thus
\[
\sigma(B) \prec_w \mu.
\]
And if $A$, $B$, and $C$ are all square of the same size, then
\[
\log |\lambda(B)| \prec_w \log \mu.
\]

**Proof.** We may assume that $B \neq 0$. Let $B = UDV^*$ be a singular value decomposition of the matrix $B$, where $D = \text{diag}(\sigma_1(B), \ldots, \sigma_r(B))$, and $U$ and $V$ are $m \times r$ and $n \times r$ partial unitary matrices, respectively, i.e., $U^*U = V^*V = I_r$. Then
\[
\begin{pmatrix}
  U^* & 0 \\
  0 & V^*
\end{pmatrix}
\begin{pmatrix}
  A & B \\
  B^* & C
\end{pmatrix}
\begin{pmatrix}
  U & 0 \\
  0 & V
\end{pmatrix}
= \begin{pmatrix}
  U^*AU & D \\
  D & V^*CV
\end{pmatrix} \geq 0.
\]
Taking the leading principal \( k \times k \) submatrix of each block, \( 1 \leq k \leq r \), we have

\[
\begin{pmatrix}
[U^*AU]_k & [D]_k \\
[D]_k & [V^*CV]_k
\end{pmatrix} \geq 0.
\]

It follows that, by taking determinant for each block,

\[ \det[D]_k^2 \leq \det([U^*AU]_k) \det([V^*CV]_k). \]

Or equivalently, for each \( 1 \leq k \leq r \),

\[
\prod_{i=1}^{k} \sigma_i(B)^2 \leq \prod_{i=1}^{k} \lambda_i([U^*AU]_k) \lambda_i([V^*CV]_k).
\]

By the eigenvalue interlacing theorem (see, e.g., [17, p. 222–224]), we arrive at

\[
\prod_{i=1}^{k} \sigma_i(B)^2 \leq \prod_{i=1}^{k} \lambda_i(A) \lambda_i(C) \leq \prod_{i=1}^{k} \mu_i^2.
\]

The desired inequality (1), thus (2), follows immediately by taking square roots. (3) is similarly obtained by letting \( B = WTW^* \), where \( T \) is an upper triangular matrix with diagonal entries \( \lambda_1(B), \lambda_2(B), \ldots, \lambda_n(B) \) and \( W \) is unitary. ■

**Corollary 1** Let \( A \geq 0, \ B \geq 0 \) be of size \( n \times n \). Then for any \( z \in \mathbb{C} \)

\[ \log \sigma(A - |z|B) \preceq_w \log \sigma(A + zB) \preceq_w \log \sigma(A + |z|B). \]  

**(4)**

**Proof.** For the second part, by (1), it is sufficient to notice that

\[
\begin{pmatrix}
A + |z|B & A + zB \\
A + z^*B & A + |z|B
\end{pmatrix} \geq 0;
\]

whereas the first part is proven by using the elementary inequality (see [12] or [13])

\[ |1 - |z|| \leq |1 - z| \leq 1 + |z|. \] ■

(4) is to appear in [13]. It refines the majorization inequality [12]

\[ \log \sigma(A - B) \preceq_w \log \sigma(A + B) \]

and implies the weaker inequality for unitarily invariant norms \( \| \cdot \|_{ui}[4] \)

\[ \|A - |z|B\|_{ui} \leq \|A + zB\|_{ui} \leq \|A + |z|B\|_{ui}. \]

We note that the following matrix inequalities do not hold in general:

\[ |A - |z|B| \leq |A - zB| \leq A + |z|B. \]
For a counterexample, take $z = i$, 
\[ A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}. \]
Then $\lambda(|A - B|) = (3, 3)$, $\lambda(|A + iB|) = (6.951 \cdots, 1.294 \cdots)$, and $\lambda(A + B) = (9, 1)$.

**Corollary 2** Let $A$ be any $n \times n$ complex matrix. Then
\[
\log |\lambda(A)| \prec \log \sigma(A). \tag{5}
\]

**Proof.** By (3), it is sufficient to notice that 
\[
\begin{pmatrix}
|A^*| & A \\
A^* & |A|
\end{pmatrix} \geq 0. \quad \blacksquare
\]

Inequality (5) is the well known Weyl's inequality (see, e.g., [7, p. 171]).

**Remark 2.1:** Note that $\mu$ in (1) and (3) cannot be replaced by $\sigma(A)$ or $\sigma(C)$.

**Remark 2.2:** In the proof of the theorem, we used the result [9, p. 142] that 
\[
\begin{pmatrix}
A & B^* \\
B & C
\end{pmatrix} \geq 0 \quad \Rightarrow \quad \det(B^*B) \leq \det A \det C,
\]
where $A$, $B$, and $C$ square matrices of the same size. (This does not hold in general if $B$ is rectangular.) Using this result, one can also give a very simple proof to the determinantal inequality [9, p. 144]: Let $\lambda_i$ be complex numbers and $A_i \geq 0$. Then 
\[
|\det(\lambda_1 A_1 + \cdots + \lambda_k A_k)| \leq \det(|\lambda_1|A_1 + \cdots + |\lambda_k|A_k).
\]
This is because 
\[
\sum_{i=1}^{k} \begin{pmatrix}
|\lambda_i|A_i & \lambda_iA_i \\
\lambda_iA_i & |\lambda_i|A_i
\end{pmatrix} \geq 0.
\]

3. SOME BASIC INEQUALITIES

Block matrices in the form \[
\begin{pmatrix}
H & K \\
K & H
\end{pmatrix}
\] have played a pivotal role in proving some matrix inequalities. We shall give some elementary matrix inequalities by applying a result on the block matrix to some partitioned positive semidefinite matrices and then to further derive inequalities on sum, ordinary and Hadamard products.

Let $H$ and $K$ be (complex) Hermitian matrices of the same size. Then 
\[
\begin{pmatrix}
H & K \\
K & H
\end{pmatrix} \geq 0 \quad \Leftrightarrow \quad H \geq \pm K. \tag{6}
\]
This is seen by noticing the matrix identity via nonsingular congruence (similarity)

\[
\left[ \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix} \right] \left( \begin{pmatrix} H & K \\ K & H \end{pmatrix} \right) \left[ \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \right] = \begin{pmatrix} H - K & 0 \\ 0 & H + K \end{pmatrix}.
\]

Obviously the eigenvalues of the block matrix in (6) are those of \( H \pm K \). A proof of (6) for the real case is given in [5] via quadratic forms, and a characterization of the matrices \( K \), which comprise a convex set, for the given \( H \) by trace inequalities is presented in [2]. A majorization inequality of the eigenvalues of the matrices \( H \) and \( K \) in (6) is seen in [14]:

\[
|\lambda(K)| \prec_w \lambda(H),
\]

which is strengthened as, by Theorem 1,

\[
\log |\lambda(K)| \prec_w \log \lambda(H),
\]

while the (stronger) eigenvalue pairwise dominant inequalities

\[
|\lambda_i(K)| \leq \lambda_i(H)
\]

do not hold for all \( i \), though \( |\lambda_1(K)| \leq \lambda_1(H) \). (Thus \( H \geq \pm K \nRightarrow H \geq |K| \).

Moreover, by using the block matrix in (6) and the Albert theorem [1], one has

\[
H \geq \pm K \Rightarrow K = HH^+K = KH^+H,
\]

where \( H^+ \) is the Moore-Penrose generalized inverse of \( H \).

We now give our basic inequalities that easily follow from (6).

**Theorem 2** Let \( A, B, \) and \( C \) be \( n \)-square complex matrices such that

\[
\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.
\]

Then, with \( \star \) for + or \( \circ \),

\[
A \star C \geq \pm (B^* \star B)
\]

and, if \( AB = BA \),

\[
A^\dagger C A^\dagger \geq B^* B.
\]

**Proof.** Since the block matrix via a permutation congruence is also positive semidefinite, we have (by the Schur Hadamard product theorem; see, e.g., [17, p. 192])

\[
\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \star \begin{pmatrix} C & B^* \\ B & A \end{pmatrix} = \begin{pmatrix} A \star C & B^* \star B \\ B^* \star B & A \star C \end{pmatrix} \geq 0.
\]

(9) thus follows from (6). For (10), notice that if \( B \) commutes with \( A \), then \( B \) commutes with \( A^\dagger \) (see, e.g., [6, p. 322] or [17, p. 165]). Let \( A \) be nonsingular. Then

\[
C \geq B^* A^{-1} B = B^* A^{-\frac{1}{2}} A^{-\frac{1}{2}} B = A^{-\frac{1}{2}} B^* B A^{-\frac{1}{2}},
\]

5
from which, by pre- and post-multiplying both sides by $A^{\frac{1}{2}}$, we arrive at the desired inequality. The singular case of $A$ follows from a continuity argument. ■

With the assumption of the theorem, we note the following.

**Remark 3.1:** For the sum in (9), the inequality $A + C \geq \pm(B + B^*)$ is also proven by observing

$$(I, \pm I) \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \begin{pmatrix} I \\ \pm I \end{pmatrix} = A \pm (B + B^*) + C \geq 0.$$ 

**Remark 3.2:** Applying (8) to (9), we have explicitly

$$\log |\lambda(B + B^*)| < w \log \lambda(A + C)$$

and

$$\log |\lambda(B \circ B^*)| < w \log \lambda(A \circ C).$$

In particular,

$$|\det(B + B^*)| \leq \det(A + C)$$

and

$$|\det(B \circ B^*)| \leq \det(A \circ C).$$

**Remark 3.3:** The condition $AB = BA$ for (10) is not removable in general, and $B$ and $B^*$ on the right hand side cannot be switched. It is easy to find an example that

$$\begin{pmatrix} I & B \\ B^* & C \end{pmatrix} \geq 0 \Rightarrow C \geq B^*B, \quad \text{but} \quad C \not\geq BB^*.$$ 

**Remark 3.4:** If the matrices $B$ and $C$ in the theorem commute, then

$$C^\frac{1}{2}AC^\frac{1}{2} \geq BB^*.$$ 

4. Applications

Applications of Theorem 2 to some frequently used $2 \times 2$ block positive semidefinite matrices result in some interesting inequalities. We present some as examples.

Assume in the following that matrices $A$, $B$, and $C$ are all $n$-square (some results also hold for the rectangular case). We itemize with the block positive semidefinite matrices followed by immediate inequalities and comments.

Inequalities of one matrix:

1. $\begin{pmatrix} A & I \\ I & A^{-1} \end{pmatrix} \geq 0$, for any $A > 0$, gives
1i). \(2I \leq A + A^{-1};\)
1ii). \(I \leq A \circ A^{-1}.\)

Comments: These are existing inequalities. Alternative proof of the first one is by a unitary diagonalization of \(A\), while the second one’s proof does not come that easy; it usually needs to prove \(A \circ A^{-1} \geq (A \circ A^{-1})^{-1}\) first (see, e.g., [8]).

If \(A\) and \(B\) are both positive definite and \(n\)-square, by noticing that

\[
\left( \begin{array}{cc} A & I \\ I & A^{-1} \end{array} \right) \circ \left( \begin{array}{cc} B^{-1} & I \\ I & B \end{array} \right) = \left( \begin{array}{cc} A \circ B^{-1} & I \\ I & A^{-1} \circ B \end{array} \right) \geq 0,
\]

we obtain a result of Visick ([10, Theorem 5 ii]):

\[
A \circ B^{-1} + B \circ A^{-1} \geq 2I.
\]

2. \(\left( \begin{array}{cc} \sigma_1 I & A^* \\ A & \sigma_1 I \end{array} \right) \geq 0\), where \(\sigma_1\) is the largest singular value of (any) \(A\), implies

2i). \(|A + A^*| \leq 2\sigma_1 I;
2ii). \(|A \circ A^*| \leq \sigma_1^2 I.

Comments: A direct proof for 2i) and 2ii) without using the theorem may not work out as smoothly, though they are weaker than the following inequality 3i).

3. \(\left( \begin{array}{cc} |A|^{2\alpha} & A^* \\ A & |A^*|^{2(1-\alpha)} \end{array} \right) \geq 0\), for any \(A\) and \(\alpha \in [0, 1]\), gives

3i). \(|A \circ A^*| \leq |A|^{2\alpha} \circ |A^*|^{2(1-\alpha)}.

Comments: Taking \(\alpha = 1\), we have the comparison of sum and ordinary product

\(|A + A^*| \leq A^*A + I\)

and the comparison of the Hadamard product and ordinary product

\(|A \circ A^*| \leq A^*A \circ I.

In particular, if \(A\) is positive semidefinite, with the above \(A\) replaced by \(A^{\frac{1}{2}}\),

\[
2A^{\frac{1}{2}} \leq A + I, \quad A^{\frac{1}{2}} \circ A^{\frac{1}{2}} \leq A \circ I = \text{diag}(A).
\]

And taking \(\alpha = \frac{1}{2}\), we have the inequalities involving sum and the two products

\(|A + A^*| \leq |A| + |A^*|, \quad |A \circ A^*| \leq |A| \circ |A^*|.

(Note: Neither \(|A + B| \leq |A| + |B|\) nor \(|A \circ B| \leq |A| \circ |B|\) holds in general.)
Inequalities of two or three matrices:

4. \[ \begin{pmatrix} A & B \\ B^* & B^*A^{-1}B \end{pmatrix} \succeq 0, \text{ for any } A > 0 \text{ and any } B, \] gives

4i). \[ \pm(B \ast B^*) \leq A \ast (B^*A^{-1}B). \]

Comments: If \( B = I \), then the block matrix is the same as the one in 1). Taking \( B = J \), the all one matrix, and switching \( A \) and \( A^{-1} \), one obtains a lower bound for the inverse of \( A \):

\[
\frac{1}{\Sigma(A)} J \leq A^{-1},
\]

where \( \Sigma(A) = \sum a_{ij} \) is the sum of all entries of \( A \). Note also that for any \( A > 0 \)

\[
\begin{pmatrix} A & J \\ J & \Sigma(A^{-1})J \end{pmatrix} \succeq 0.
\]

With a similar block matrix for \( B > 0 \), one obtains a lower bound for \( A \circ B \):

\[
A \circ B \geq \frac{1}{\Sigma(A^{-1})\Sigma(B^{-1})} J.
\]

5. \[ \begin{pmatrix} A & A^*C^*B^* \\ B^* & C^*B^*A^{-1}B \end{pmatrix} \succeq 0, \text{ for } A, B \geq 0 \text{ and any contraction matrix } C, \] gives

5i). \( (A^*C^*B^*) \ast (B^*C^*A^*) \leq A \ast B. \)

Comments: Taking \( B = C = I \) for the Hadamard product yields \( A^* \circ A \leq A \circ I \) as seen in 3). 5i) is equivalent to \( (ACB) \ast (BCA) \leq A^2 \ast B^2 \). Setting \( C = I \) gives \( AB + BA \leq A^2 + B^2 \) and its Hadamard companion \( AB \circ BA \leq A^2 \circ B^2 \).

6. \[ \begin{pmatrix} A^*A & A^*B \\ B^*A & B^*B \end{pmatrix} \succeq 0, \text{ for any } A \text{ and } B, \] gives

6i). \( \pm(A^*B \ast B^*A) \leq A^*A \ast B^*B. \)

Comments: The Hadamard product case of 6i) is seen in [10, Corollary 12]. In particular, if we take \( B = I \) for the Hadamard product, then \( \pm(A^* \circ A) \leq A^* A \circ I \). Letting \( A > 0 \) and setting \( B = A^{-1} \) results in 1ii).

7. \[ \begin{pmatrix} I + A^*A & A^* + B^* \\ A + B & I + BB^* \end{pmatrix} \succeq 0, \text{ for any } A \text{ and } B, \] gives

7i). \( (A + B) \circ (A + B)^* \leq (I + A^*A) \circ (I + BB^*). \)
Comments: This Hadamard product matrix inequality is compared to the conventional product matrix inequality (by taking Schur complement)

\[(A + B)(I + A^*A)^{-1}(A + B)^* \leq I + BB^*.
\]

8. \[
\begin{pmatrix}
AA^* \circ I & A \circ B \\
A^* \circ B^* & B^*B \circ I
\end{pmatrix} \geq 0, \text{ for any } A \text{ and } B,
\]
gives

8i). \(A \circ B + A^* \circ B^* \leq AA^* \circ I + B^*B \circ I;\)
8ii). \(A \circ A^* \circ B \circ B^* \leq AA^* \circ B^*B \circ I.
\]

Comments: For \(A \geq 0\) and \(B \geq 0\), 8i) gives the inequality of means for Hadamard product

\[A \circ B \leq \frac{A^2 + B^2}{2} \circ I.
\]

It follows that for any correlation matrices \(A\) and \(B\) (with diagonal entries 1)

\[A^\frac{1}{2} \circ B^\frac{1}{2} \leq I.
\]

Notice that \(AA^* \leq \sigma_1 I\). We put \(B = A^t\), the transpose of \(A\) in 8ii). Then

\[A \circ A^* \circ A^t \circ \overline{A} \leq \sigma_1^4 I.
\]

9. \[
\begin{pmatrix}
AA^* \circ BB^* & A \circ B \\
A^* \circ B^* & I
\end{pmatrix} \geq 0, \text{ for any } A \text{ and } B,
\]
gives

9i). \((A \circ B)(A^* \circ B^*) \leq AA^* \circ BB^*.
\]

Comments: This has appeared in [15] and in a recent paper [10, Theorem 4].

10. \[
\begin{pmatrix}
|A| \circ |B| & A^* \circ B^* \\
A \circ B & |A^*| \circ |B^*|
\end{pmatrix} \geq 0, \text{ for any } A \text{ and } B,
\]
gives

10i). \(|A \circ B + A^* \circ B^*| \leq |A| \circ |B| + |A^*| \circ |B^*|;\)
10ii). \(|A \circ A^* \circ B \circ B^*| \leq |A| \circ |A^*| \circ |B| \circ |B^*|.
\]

Comments: By taking \(B\) to be a matrix of 0 and 1, one can get the inequalities for the specified entries of \(A\). For example, if \(B\) is a permutation matrix, then \(|B| = |B^*| = I\) and one gets the inequalities that compare any diagonal (entries) of \(A\) to the diagonals of \(|A|\) and \(|A^*|\). And one may also obtain inequalities for submatrices of \(A\) by setting \(B = \left(\begin{array}{cc} 0 & 0 \\ J & 0 \end{array}\right)\).
11. \( \begin{pmatrix} A & AB \\ B^*A & B^*AB \end{pmatrix} \geq 0 \), for any \( A \geq 0 \) and any \( n \times m \) matrix \( B \), implies

11i). \( B^*A + AB \leq A + B^*AB; \)
11ii). \( B^*A \circ AB \leq A \circ (B^*AB). \)

Comments: Setting \( B = A^k \) yields the inequalities of shifting \( A \)
\[ 2A^{k+1} \leq A + A^{2k+1}, \quad k = 1, 2, \ldots, \]
and
\[ A^{k+1} \circ A^{k+1} \leq A \circ A^{2k+1}, \quad k = 1, 2, \ldots. \]

Inequalities of generalized inverses:

12. \( \begin{pmatrix} A & AA^+ \\ A^+A & A^+ \end{pmatrix} \geq 0 \), for any \( A \geq 0 \), gives

12i). \( A \circ A^+ \geq A^+A \circ AA^+; \)
12ii). \( A + A^+ \geq A^+A + AA^+. \)

Comments: These are compared to the inequality of Visick in [11, p. 282]:
\[ A \circ A^+ \geq (AA^+ \circ AA^+)(A \circ A^+)^+(AA^+ \circ AA^+). \]

Combining the above block matrices via sum or Hadamard product, one may get more block positive semidefinite matrices and thus more inequalities. For instance, if \( A \geq 0 \) and \( B > 0 \), both \( n \)-square, then
\[ \begin{pmatrix} I & A \\ A & A^2 \end{pmatrix} \circ \begin{pmatrix} B & I \\ I & B^{-1} \end{pmatrix} = \begin{pmatrix} I \circ B & I \circ A \\ I \circ A & I \circ A^2 \circ B^{-1} \end{pmatrix} \geq 0. \]

Thus
\[ A^2 \circ B^{-1} \geq (I \circ A)(I \circ B)^{-1}(I \circ A) = (\text{diag } A)^2(\text{diag } B)^{-1}. \]

Note that the right hand side involves only the diagonal entries of \( A \) and \( B \). In addition, for any correlation matrix \( A \) and nonsingular correlation matrix \( B \)
\[ A^2 \circ B^{-1} \geq I. \]

More inequalities are available by substituting the above matrices with matrices involving Kronecker product and by using the fact that the Hadamard product is a principal submatrix of the Kronecker product when the matrices are square. One also gets majorization inequalities by applying Theorem 1 to the above block matrices.
References


